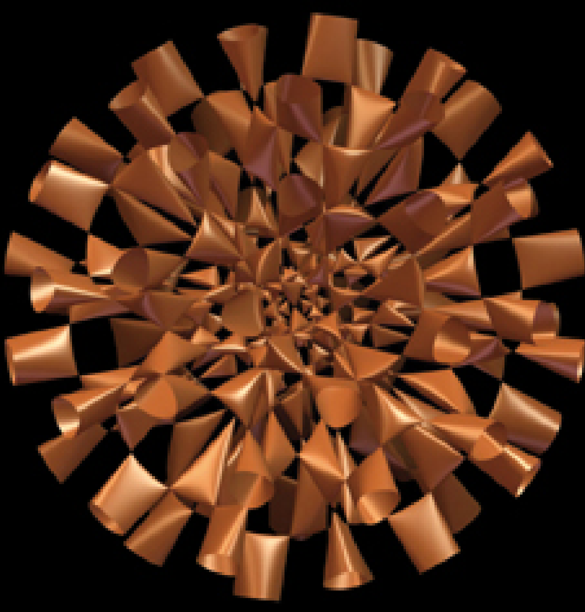
Mathematical Olympiad in China (2007–2008)



Problems and Solutions

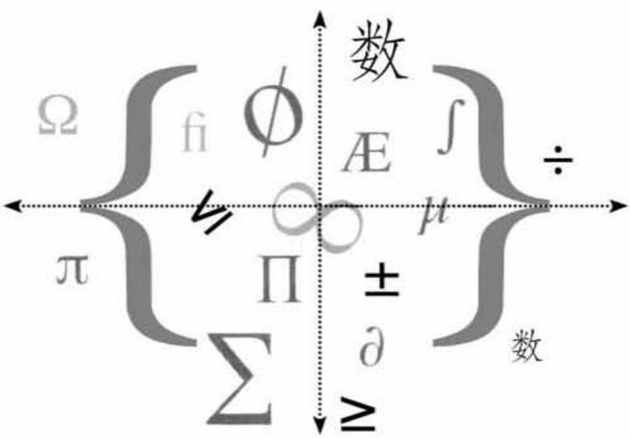
Xiong Bin | Lee Peng Yee Editors

Mathematical Olympiad in China (2007—2008)

Problems and Solutions

Mathematical Olympiad in China (2007—2008)

Problems and Solutions



Editors

Xiong Bin

East China Normal University, China

Lee Peng Yee

Nanyang Technological University, Singapore

East China Normal University Press

World Scientific

Published by

East China Normal University Press
3663 North Zhongshan Road
Shanghai 200062
China

and

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

MATHEMATICAL OLYMPIAD IN CHINA (2007–2008)
Problems and Solutions

ISBN-13 978-981-4261-14-2 (pbk)
ISBN-10 981-4261-14-9 (pbk)

Printed in Singapore.

Editors

XIONG Bin *East China Normal University, China*

Lee Peng Yee *Nanyang Technological University*

Original Authors

MO Chinese National Coaches of 2007—2008

English Translators

XIONG Bin *East China Normal University, China*

FENG Zhigang *Shanghai High School, China*

WANG Shanping *East China Normal University, China*

ZHENG Zhongyi *High School Affiliated to Fudan University, China*

ZHAO Yingting *East China Normal University, China*

Copy Editors

NI Ming *East China Normal University press, China*

ZHANG Ji *World Scientific Publishing Co., Singapore*

WONG Fook Sung *Temasek Polytechnic, Singapore*

KONG Lingzhi *East China Normal University press, China*

Preface

The first time China sent a team to IMO was in 1985. At the time, two students were sent to take part in the 26th IMO. Since 1986, China always sent a team of 6 students to IMO except in 1998 when it was held in Taiwan. So far up to 2008, China has achieved the number one ranking in team effort 14 times. A great majority of students received gold medals. The fact that China obtained such encouraging result is due to, on one hand, Chinese students' hard working and perseverance, and on the other hand, the effort of teachers in schools and the training offered by national coaches. As we believe, it is also a result of the educational system in China, in particular, the emphasis on training of basic skills in science education.

The materials of this book come from a series of two books (in Chinese) on *Forward to* IMO: *a collection of mathematical Olympiad problems* (2007 – 2008). It is a collection of problems and solutions of the major mathematical competitions in China. It provides a glimpse of how the China national team is selected and formed. First, there is the China Mathematical Competition, a national event. It is held on the second Sunday of October every year. Through the competition, about 150 students are selected to join the China Mathematical Olympiad (commonly known as the winter camp), or in short CMO, in

January of the second year. CMO lasts for five days. Both the type and the difficulty of the problems match those of IMO. Similarly, the student contestants are requested to solve three problems every day in 4.5 hours. From CMO about 20 to 30 students are selected to form a national training team. The training takes place for two weeks in the month of March every year. After six to eight tests, plus two qualifying examinations, six students are finally selected to form the national team, taking part in IMO in July of that year.

In view of the differences in education, culture and economy of West China in comparison with East China, mathematical competitions in West China did not develop as fast as in the east. In order to promote the activity of mathematical competition, and to upgrade the level of mathematical competition, starting from 2001 China Mathematical Olympiad Committee conducted the China Western Mathematical Olympiad. The top two winners will be admitted to the national training team. Through the China Western Mathematical Olympiad, there have been two students entering the national team and receiving gold medals for their performance at IMO.

So far since 1986, the China team has never had a female student. In order to encourage more female students participating in mathematical competitions, starting from 2002 China Mathematical Olympiad Committee conducted the China Girls' mathematical Olympiad. Again, the top two winners will be admitted directly into the national training team. In 2007, the first girl who was winner of China Girls' mathematical Olympiad was selected to enter the 2008 China national team and won a gold medal of the 49th IMO.

The authors of this book are coaches of the MO Chinese national team. They are Xiong Bin, Li Shenghong, Leng Gangsong, Wu Jianping, Chen Yonggao, Wang Jianwei, Li Weigu, Yu Hongbing, Zhu Huawei, Feng Zhigang, Liu Shixiong, Zhang Sihu, and Zheng Chongyi. Those who took part in the translation work are Xiong Bin, Feng Zhigang, Wang Shanping, Zheng Chongyi, and Zhao Yingting. We are grateful to Qiu Zhonghu, Wang Jie, Wu Jianping, and Pan Chengbiao for their guidance and assistance to authors. We are grateful to Ni Ming of East China Normal University Press. Their effort has helped make our job easier. We are also grateful to Zhang Ji of World Scientific Publishing for her hard work leading to the final publication of the book.

Authors
October 2008

This page intentionally left blank

Introduction

Early days

The International Mathematical Olympiad (IMO), founded in 1959, is one of the most competitive and highly intellectual activities in the world for high school students.

Even before IMO, there were already many countries which had mathematics competition. They were mainly the countries in Eastern Europe and in Asia. In addition to the popularization of mathematics and the convergence in educational systems among different countries, the success of mathematical competitions at the national level provided a foundation for the setting-up of IMO. The countries that asserted great influence are Hungary, the former Soviet Union and the United States. Here is a brief review of the IMO and mathematical competitions in China.

In 1894, the Department of Education in Hungary passed a motion and decided to conduct a mathematical competition for the secondary schools. The well-known scientist, *J. von Etövös*, was the Minister of Education at that time. His support in the event had made it a success and thus it was well publicized. In addition, the success of his son, *R. von Etövös*, who was also a physicist, in proving the principle of equivalence of the general theory of relativity by *A. Einstein* through experiment, had brought Hungary to the world stage in science. Thereafter, the prize for the mathematical competition in Hungary was named "*Etövös* prize". This was the first formally organized mathematical competition in the world. In what follows, Hungary had indeed produced a lot of well-known scientists including *L. Fejér*, *G. Szegö*, *T. Radó*, *A. Haar* and *M. Riesz* (in real analysis), *D. König* (in combinatorics), *T. von Kármán* (in aerodynamics), and *J. C. Harsanyi* (in game theory), who had also won the Nobel Prize for Economics in 1994. They all were the winners of Hungary mathematical competition. The top scientific genius of Hungary, *J. von Neumann*, was one of the leading mathematicians in the 20th century. *Neumann* was overseas while the competition took place. Later he did it himself and it took him half an hour to complete. Another mathematician worth mentioning is the highly productive number theorist *P. Erdös*. He was a pupil of *Fejér* and also a winner of the Wolf Prize. *Erdös* was very passionate about mathematical competitions and setting competition questions. His contribution to discrete mathematics was unique and greatly significant. The rapid progress and development of discrete mathematics over the subsequent decades had indirectly influenced the types of questions set in IMO. An internationally recognized prize named after Erdös was to honour those who had contributed to the education of mathematical competitions. Professor *Qiu Zonghu* from China had won the prize in 1993.

In 1934, a famous mathematician *B. Delone* conducted a mathematical competition for high school students in Leningrad (now St. Petersburg) in formal USSR. In 1935, Moscow also started organizing such event. Other than being interrupted during the World War II, these events had been carried on until today. As for the Russian Mathematical Competition (later renamed as the Soviet Mathematical Competition), it was not started until 1961. Thus, the former Soviet Union and Russia became the leading powers of Mathematical Olympiad. A lot of grandmasters in mathematics including the great *A. N. Kolmogorov* were all very enthusiastic about the mathematical competition. They would personally involve in setting the questions for the competition. The former Soviet Union even called it the Mathematical Olympiad, believing that mathematics is the "gymnastics of thinking". These points of view gave a great impact on the educational community. The winner of the Fields Medal in 1998, *M. Kontsevich*, was once the first runner-up of the Russian Mathematical Competition. *G. Kasparov*, the international chess grandmaster, was once the second runner-up. *Grigori Perelman*, the winner of the Fields Medal in 2006, who solved the final step of Poincaré's Conjecture, was a gold medalist of IMO in 1982.

In the United States of America, due to the active promotion by the renowned mathematician *Birkhoff* and his son, together with *G. Pólya*, the Putnam mathematics competition was organized in 1938 for junior undergraduates. Many of the questions were within the scope of high school students. The top five contestants of the Putnam mathematical competition would be entitled to the membership of Putnam. Many of these were eventually outstanding mathematicians. There were the famous *R. Feynman* (winner of the Nobel Prize for Physics, 1965), *K. Wilson* (winner of the Nobel Prize for Physics, 1982), *J. Milnor* (winner of the Fields Medal, 1962), *D. Mumford* (winner of the Fields Medal, 1974), and *D.*

Quillen (winner of the Fields Medal, 1978).

Since 1972, in order to prepare for the IMO, the United States of America Mathematical Olympiad (USAMO) was organized. The standard of questions posed was very high, parallel to that of the Winter Camp in China. Prior to this, the United States had organized American High School Mathematics Examination (AHSME) for the high school students since 1950. This was at the junior level yet the most popular mathematics competition in America. Originally, it was planned to select about 100 contestants from AHSME to participate in USAMO. However, due to the discrepancy in the level of difficulty between the two competitions and other restrictions, from 1983 onwards, an intermediate level of competition, namely, American Invitational Mathematics Examination (AIME), was introduced. Henceforth both AHSME and AIME became internationally well-known. A few cities in China had participated in the competition and the results were encouraging.

The members of the national team who were selected from USAMO would undergo training at the West Point Military Academy, and would meet the President at the White House together with their parents. Similarly as in the former Soviet Union, the Mathematical Olympiad education was widely recognized in America. The book "How to Solve it" written by *George Polya* along with many other titles had been translated into many different languages. *George Polya* provided a whole series of general heuristics for solving problems of all kinds. His influence in the educational community in China should not be underestimated.

International Mathematical Olympiad

In 1956, the East European countries and the Soviet Union took the initiative to organize the IMO formally. The first International Mathematical Olympiad (IMO) was held in Brasov, Romania, in 1959. At the time, there were only seven participating countries, namely, Romania, Bulgaria, Poland, Hungary, Czechoslovakia, East Germany and the Soviet Union. Subsequently, the United States of America, United Kingdom, France, Germany and also other countries including those from Asia joined. Today, the IMO had managed to reach almost all the developed and developing countries. Except in the year 1980 due to financial difficulties faced by the host country, Mongolia, there were already 49 Olympiads held and 97 countries participating.

The mathematical topics in the IMO include number theory, polynomials, functional equations, inequalities, graph theory, complex numbers, combinatorics, geometry and game theory. These areas had provided guidance for setting questions for the competitions. Other than the first few Olympiads, each IMO is normally held in mid-July every year and the test paper consists of 6 questions in all. The actual competition lasts for 2 days for a total of 9 hours where participants are required to complete 3 questions each day. Each question is 7 marks which total up to 42 marks. The full score for a team is 252 marks. About half of the participants will be awarded a medal, where 1/12 will be awarded a gold medal. The numbers of gold, silver and bronze medals awarded are in the ratio of 1 : 2 : 3 approximately. In the case when a participant provides a better solution than the official answer, a special award is given.

Each participating country will take turn to host the IMO.

The cost is borne by the host country. China had successfully hosted the 31st IMO in Beijing. The event had made a great impact on the mathematical community in China. According to the rules and regulations of the IMO, all participating countries are required to send a delegation consisting of a leader, a deputy leader and 6 contestants. The problems are contributed by the participating countries and are later selected carefully by the host country for submission to the international jury set up by the host country. Eventually, only 6 problems will be accepted for use in the competition. The host country does not provide any question. The short-listed problems are subsequently translated, if necessary, in English, French, German, Russian and other working languages. After that, the team leaders will translate the problems into their own languages.

The answer scripts of each participating team will be marked by the team leader and the deputy leader. The team leader will later present the scripts of their contestants to the coordinators for assessment. If there is any dispute, the matter will be settled by the jury. The jury is formed by the various team leaders and an appointed chairman by the host country. The jury is responsible for deciding the final 6 problems for the competition. Their duties also include finalizing the marking standard, ensuring the accuracy of the translation of the problems, standardizing replies to written queries raised by participants during the competition, synchronizing differences in marking between the leaders and the coordinators and also deciding on the cut-off points for the medals depending on the contestants' results as the difficulties of problems each year are different.

China had participated informally in the 26th IMO in 1985. Only two students were sent. Starting from 1986, except in 1998 when the IMO was held in Taiwan, China had always sent 6 official contestants to the IMO. Today, the Chinese contestants not only performed outstandingly in the IMO, but also in the International Physics, Chemistry, Informatics, and Biology Olympiads. So far, no other countries have overtaken China in the number of gold and silver medals received. This can be regarded as an indication that China pays great attention to the training of basic skills in mathematics and science education.

Winners of the IMO

Among all the IMO medalists, there were many who eventually became great mathematicians. They were also awarded the Fields Medal, Wolf Prize and Nevanlinna Prize (a prominent mathematics prize for computing and informatics). In what follows, we name some of the winners.

G. Margulis, a silver medalist of IMO in 1959, was awarded the Fields Medal in 1978. *L. Lovasz*, who won the Wolf Prize in 1999, was awarded the Special Award in IMO consecutively in 1965 and 1966. *V. Drinfeld*, a gold medalist of IMO in 1969, was awarded the Fields Medal in 1990. *J.-C. Yoccoz* and *T. Gowers*, who were both awarded the Fields Medal in 1998, were gold medalists in IMO in 1974 and 1981 respectively. A silver medalist of IMO in 1985, *L. Lafforgue*, won the Fields Medal in 2002. A gold medalist of IMO in 1982, *Grigori Perelman* from Russia, was awarded the Fields Medal in 2006 for solving the final step of the Poincaré conjecture. In 1986, 1987, and 1988, *Terence Tao* won a bronze, silver, and gold medal respectively. He was the youngest participant to date in the IMO, first competing at the age of ten. He was also awarded the Fields Medal in 2006.

A silver medalist of IMO in 1977, *P. Shor*, was awarded the Nevanlinna Prize. A gold medalist of IMO in 1979, *A. Razborov*, was awarded the Nevanlinna Prize. Another gold medalist of IMO in 1986, *S. Smirnov*, was awarded the Clay Research Award. *V. Lafforgue*, a gold medalist of IMO in 1990, was awarded the European Mathematical Society prize. He is *L. Laforgue's* younger brother.

Also, a famous mathematician in number theory, *N. Elkis*, who is also a foundation professor at Havard University, was awarded a gold medal of IMO in 1981. Other winners include *P. Kronheimer* awarded a silver medal in 1981 and *R. Taylor* a contestant of IMO in 1980.

Mathematical competition in China

Due to various reasons, mathematical competition in China started relatively late but is progressing vigorously.

"We are going to have our own mathematical competition too!" said *Hua Luogeng*. *Hua* is a house-hold name in China. The first mathematical competition was held concurrently in Beijing, Tianjing, Shanghai and Wuhan in 1956. Due to the political situation at the time, this event was interrupted a few times. Until 1962, when the political environment started to improve, Beijing and other cities started organizing the competition though not regularly. In the era of cultural revolution, the whole educational system in China was in chaos. The mathematical competition came to a complete halt. In contrast, the mathematical competition in the former Soviet

Union was still on-going during the war and at a time under the difficult political situation. The competitions in Moscow were interrupted only 3 times between 1942 and 1944. It was indeed commendable.

In 1978, it was the spring of science. *Hua Luogeng* conducted the Middle School Mathematical Competition for 8 provinces in China. The mathematical competition in China was then making a fresh start and embarked on a road of rapid development. *Hua* passed away in 1985. In commemorating him, a competition named *Hua Luogeng* Gold Cup was set up in 1986 for the junior middle school students and it had a great impact.

The mathematical competitions in China before 1980 can be considered as the initial period. The problems set were within the scope of middle school textbooks. After 1980, the competitions were gradually moving towards the senior middle school level. In 1981, the Chinese Mathematical Society decided to conduct the China Mathematical Competition, a national event for high schools.

In 1981, the United States of America, the host country of IMO, issued an invitation to China to participate in the event. Only in 1985, China sent two contestants to participate informally in the IMO. The results were not encouraging. In view of this, another activity called the Winter Camp was conducted after the China Mathrmatifcal Competition. The Winter Camp was later renamed as the China Mathematical Olympiad or CMO. The winning team would be awarded the *Chern Shiing-Shen* Cup. Based on the outcome at the Winter Camp, a selection would be made to form the 6 - member national team for IMO. From 1986 onwards, other than the year when IMO was organized in Taiwan, China had been sending a 6 - member team to IMO. China is normally awarded the champion or first runner-up except on three occasions when the results were lacking. Up to 2006, China had been awarded the overall team champion for 13 times.

In 1990, China had successfully hosted the 31st IMO. It showed that the standard of mathematical competition in China has leveled that of other leading countries. First, the fact that China achieves the highest marks at the 31st IMO for the team is an evidence of the effectiveness of the pyramid approach in selecting the contestants in China. Secondly, the Chinese mathematicians had simplified and modified over 100 problems and submitted them to the team leaders of the 35 countries for their perusal. Eventually, 28 problems were recommended. At the end, 5 problems were chosen (IMO requires 6 problems). This is another evidence to show that China has achieved the highest quality in setting problems. Thirdly, the answer scripts of the participants were marked by the various team leaders and assessed by the coordinators who were nominated by the host countries. China had formed a group 50 mathematicians to serve as coordinators who would ensure the high accuracy and fairness in marking. The marking process was completed half a day earlier than it was scheduled. Fourthly, that was the first ever IMO organized in Asia. The outstanding performance by China had encouraged the other developing countries, especially those in Asia. The organizing and coordinating work of the IMO by the host country was also reasonably good.

In China, the outstanding performance in mathematical competition is a result of many contributions from the all quarters of mathematical community. There are the older generation of mathematicians, middle-aged mathematicians and also the middle and elementary school teachers. There is one person who deserves a special mention and he is *Hua Luogeng*. He initiated and promoted the mathematical competition. He is also the author of the following books: Beyond *Yang hui*'s Triangle, Beyond the *pi* of *Zu Chongzhi*, Beyond the Magic Computation of *Sun-zi*, Mathematical Induction, and Mathematical Problems of Bee Hive. These were his books derived from mathematics competitions. When China resumed mathematical competition in 1978, he participated in setting problems and giving critique to solutions of the problems. Other outstanding books derived from the Chinese mathematics competitions are: Symmetry by *Duan Xuefu*, Lattice and Area by *Min Sihe*, One Stroke Drawing and Postman Problem by *Jiang Boju*.

After 1980, the younger mathematicians in China had taken over from the older generation of mathematicians in running the mathematical competition. They worked and strived hard to bring the level of mathematical competition in China to a new height. *Qiu Zonghu* is one such outstanding representative. From the training of contestants and leading the team 3 times to IMO to the organizing of the 31th IMO in China, he had contributed prominently and was awarded the *P. Erdös* prize.

Preparation for IMO

Currently, the selection process of participants for IMO in China is as follows.

First, the China Mathematical Competition, a national competition for high Schools, is organized on the second Sunday in October every year. The objectives are: to increase the interest of students in learning mathematics, to promote the development of co-curricular activities in mathematics, to help improve the teaching of mathematics in high schools, to discover and cultivate the talents and also to prepare for the IMO. This happens since 1981. Currently there are about 200,000 participants taking part.

Through the China Mathematical Competition, around 150 of students are selected to take part in the China Mathematical Olympiad or CMO, that is, the Winter Camp. The CMO lasts for 5 days and is held in January every year. The types and difficulties of the problems in CMO are very much similar to the IMO. There are also 3 problems to be completed within 4.5 hours each day. However, the score for each problem is 21 marks which add up to 126 marks in total. Starting from 1990, the Winter Camp instituted the *Chern Shiing-Shen* Cup for team championship. In 1991, the Winter Camp was officially renamed as the China Mathematical Olympiad (CMO). It is similar to the highest national mathematical competition in the former Soviet Union and the United States.

The CMO awards the first, second and third prizes. Among the participants of CMO, about 20 to 30 students are selected to participate in the training for IMO. The training takes place in March every year. After 6 to 8 tests and another 2 rounds of qualifying examinations, only 6 contestants are short-listed to form the China IMO national team to take part in the IMO in July.

Besides the China Mathematical Competition (for high schools), the Junior Middle School Mathematical Competition is also developing well. Starting from 1984, the competition is organized in April every year by the Popularization Committee of the Chinese Mathematical Society. The various provinces, cities and autonomous regions would rotate to host the event. Another mathematical competition for the junior middle schools is also conducted in April every year by the Middle School Mathematics Education Society of the Chinese Educational Society since 1998 till now.

The *Hua Luogeng* Gold Cup, a competition by invitation, had also been successfully conducted since 1986. The participating students comprise elementary six and junior middle one students. The format of the competition consists of a preliminary round, semi-finals in various provinces, cities and autonomous regions, then the finals.

Mathematical competition in China provides a platform for students to showcase their talents in mathematics. It encourages learning of mathematics among students. It helps identify talented students and to provide them with differentiated learning opportunity. It develops co-curricular activities in mathematics. Finally, it brings about changes in the teaching of mathematics.

Contents

China Mathematical Competition

2006 (Zhejiang)

Popularization Committee of Chinese Mathematical Society and Zhejiang Mathematical Society were responsible for the assignment of the competition problems in the first round and the extra round of the competition.

Part I Multiple-choice Questions (Questions 1 to 6, each carries 6 marks.)

1 Let $\triangle ABC$ be a given triangle. If $|\overrightarrow{BA} - t\,\overrightarrow{BC}| \geqslant |\overrightarrow{AC}|$ for any $t \in \mathbf{R}$, then $\triangle ABC$ is (   ).

(A) an acute triangle (B) an obtuse triangle

(C) a right triangle (D) not known

Solution Suppose $\angle ABC = \alpha$. Since $|\overrightarrow{BA} - t\overrightarrow{BC}| \geqslant |\overrightarrow{AC}|$, we have

$$|\overrightarrow{BA}|^2 - 2t\overrightarrow{BA} \cdot \overrightarrow{BC} + t^2|\overrightarrow{BC}|^2 \geqslant |\overrightarrow{AC}|^2.$$

Let

$$t = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BC}|^2},$$

we get

$$|\overrightarrow{BA}|^2 - 2|\overrightarrow{BA}|^2\cos^2\alpha + \cos^2\alpha|\overrightarrow{BA}|^2 \geqslant |\overrightarrow{AC}|^2.$$

That means $|\overrightarrow{BA}|^2\sin^2\alpha \geqslant |\overrightarrow{AC}|^2$, i.e. $|\overrightarrow{BA}|\sin\alpha \geqslant |\overrightarrow{AC}|$.

On the other hand, let point D lie on line BC such that $AD \perp BC$. Then we have $|\overrightarrow{BA}|\sin\alpha = |\overrightarrow{AD}| \leqslant |\overrightarrow{AC}|$. Hence $|\overrightarrow{AD}| = |\overrightarrow{AC}|$, and that means $\angle ACB = \frac{\pi}{2}$. Answer: C.

2 Suppose $\log_x(2x^2 + x - 1) > \log_x 2 - 1$. Then the range of x is ( ).

(A) $\frac{1}{2} < x < 1$ (B) $x > \frac{1}{2}$ and $x \neq 1$

(C) $x > 1$ (D) $0 < x < 1$

Solution From $\begin{cases} x > 0, \\ x \neq 1, \\ 2x^2 + x - 1 > 0, \end{cases}$ we get $x > \frac{1}{2}$, $x \neq 1$.

Furthermore,

$\log_x(2x^2 + x - 1) > \log_x 2 - 1 \Rightarrow \log_x(2x^3 + x^2 - x) > \log_x 2 \Rightarrow$ $\begin{cases} 0 < x < 1, \\ 2x^3 + x^2 - x < 2, \end{cases}$ or $\begin{cases} x > 1, \\ 2x^3 + x^2 - x > 2. \end{cases}$ Then we have

$x > \dfrac{1}{2}$ and $x \neq 1$. Answer: B.

3 Suppose $A = \{x \mid 5x - a \leqslant 0\}$, $B = \{x \mid 6x - b > 0\}$, $a, b \in \mathbf{N}$, and $A \cap B \cap \mathbf{N} = \{2, 3, 4\}$. The number of such pairs (a, b) is ( ).

(A) 20 (B) 25 (C) 30 (D) 42

Solution Since $5x - a \leqslant 0 \Rightarrow x \leqslant \dfrac{a}{5}$, $6x - b > 0 \Rightarrow x > \dfrac{b}{6}$. In order to satisfy $A \cap B \cap \mathbf{N} = \{2, 3, 4\}$, we have $\begin{cases} 1 \leqslant \dfrac{b}{6} < 2, \\ 4 \leqslant \dfrac{a}{5} < 5, \end{cases}$ or $\begin{cases} 6 \leqslant b < 12, \\ 20 \leqslant a < 25. \end{cases}$ So the number of pairs (a, b) is $\binom{6}{1} \cdot \binom{5}{1} = 30$. Answer: C.

4 Given a right triangular prism $A_1B_1C_1 - ABC$ with $\angle BAC = \dfrac{\pi}{2}$ and $AB = AC = AA_1 = 1$, let G, E be the midpoints of A_1B_1, CC_1 respectively; and D, F be variable points lying on segments AC, AB (not including endpoints) respectively. If $GD \perp EF$, the range of the length of DF is ( ).

(A) $\left[\dfrac{1}{\sqrt{5}}, 1\right)$ (B) $\left[\dfrac{1}{5}, 2\right)$

(C) $[1, \sqrt{2})$ (D) $\left[\dfrac{1}{\sqrt{5}}, \sqrt{2}\right)$

Solution We establish a coordinate system with point A as the origin, line AB as the x-axis, AC the y-axis and AA_1 the z-axis. Then we have $F(t_1, 0, 0)$ $(0 < t_1 < 1)$, $E\left(0, 1, \dfrac{1}{2}\right)$,

$G\left(\frac{1}{2}, 0, 1\right)$, $D(0, t_2, 0)$ $(0 < t_2 < 1)$. Therefore $\overrightarrow{EF} = \left(t_1, -1, -\frac{1}{2}\right)$, $\overrightarrow{GD} = \left(-\frac{1}{2}, t_2, -1\right)$. Since $GD \perp EF$, we get $t_1 + 2t_2 = 1$. Then $0 < t_2 < \frac{1}{2}$. Furthermore, $\overrightarrow{DF} = (t_1, -t_2, 0)$,

$$|\overrightarrow{DF}| = \sqrt{t_1^2 + t_2^2} = \sqrt{5t_2^2 - 4t_2 + 1} = \sqrt{5\left(t_2 - \frac{2}{5}\right)^2 + \frac{1}{5}}.$$

We obtain $\sqrt{\frac{1}{5}} \leqslant |\overrightarrow{DF}| < 1$. Answer: A.

5 Suppose $f(x) = x^3 + \log_2(x + \sqrt{x^2 + 1})$. For any $a, b \in \mathbf{R}$, to satisfy $f(a) + f(b) \geqslant 0$, the condition $a + b \geqslant 0$ is ( ).

(A) necessary and sufficient

(B) not necessary but sufficient

(C) necessary but not sufficient

(D) neither necessary nor sufficient

Solution Obviously $f(x) = x^3 + \log_2(x + \sqrt{x^2 + 1})$ is an odd function and is monotonically increasing. So, if $a + b \geqslant 0$, i.e. $a \geqslant -b$, we get $f(a) \geqslant f(-b)$, $f(a) \geqslant -f(b)$, and that means $f(a) + f(b) \geqslant 0$.

On the other hand, if $f(a) + f(b) \geqslant 0$, then $f(a) \geqslant -f(b) = f(-b)$. So $a \geqslant -b$, $a + b \geqslant 0$. Answer: A.

6 Let S be the set of all those 2007-place decimal integers $\overline{2a_1a_2a_3 \cdots a_{2006}}$ which contain odd number of digit '9' in each sequence $a_1, a_2, a_3, \cdots, a_{2006}$. The cardinal number of S is

(A) $\frac{1}{2}(10^{2006}+8^{2006})$ (B) $\frac{1}{2}(10^{2006}-8^{2006})$

(C) $10^{2006}+8^{2006}$ (D) $10^{2006}-8^{2006}$

Solution Define A as the number of the elements in S, we have

$$A=\binom{2006}{1}9^{2005}+\binom{2006}{3}9^{2003}+\cdots+\binom{2006}{2005}9.$$

On the other hand,

$$(9+1)^{2006}=\sum_{k=0}^{2006}\binom{2006}{k}9^{2006-k}$$

and

$$(9-1)^{2006}=\sum_{k=0}^{2006}\binom{2006}{k}(-1)^k9^{2006-k}.$$

So

$$\begin{aligned}A&=\binom{2006}{1}9^{2005}+\binom{2006}{3}9^{2003}+\cdots+\binom{2006}{2005}9\\&=\frac{1}{2}(10^{2006}-8^{2006}).\end{aligned}$$

Answer: B.

Part II Short-Answer Questions (Questions 7 to 12, each carries 9 marks.)

7 Let $f(x)=\sin^4x-\sin x\cos x+\cos^4x$, the range of $f(x)$ is ________.

Solution As

$$\begin{aligned}f(x)&=\sin^4x-\sin x\cos x+\cos^4x\\&=1-\frac{1}{2}\sin 2x-\frac{1}{2}\sin^2 2x,\end{aligned}$$

we define $t = \sin 2x$, then

$$f(x) = g(t) = 1 - \frac{1}{2}t - \frac{1}{2}t^2 = \frac{9}{8} - \frac{1}{2}\left(t + \frac{1}{2}\right)^2.$$

So we have

$$\min_{-1 \leqslant t \leqslant 1} g(t) = g(1) = \frac{9}{8} - \frac{1}{2} \times \frac{9}{4} = 0,$$

and

$$\max_{-1 \leqslant t \leqslant 1} g(t) = g\left(-\frac{1}{2}\right) = \frac{9}{8} - \frac{1}{2} \times 0 = \frac{9}{8}.$$

Hence $0 \leqslant f(x) \leqslant \frac{9}{8}$.

8 Let complex number $z = (a + \cos\theta) + (2a - \sin\theta)\mathrm{i}$. If $|z| \leqslant 2$ for any $\theta \in \mathbf{R}$, then the range of real number a is ________.

Solution By the definition given above, we have, for any $\theta \in \mathbf{R}$,

$$\begin{aligned} |z| \leqslant 2 &\Leftrightarrow (a + \cos\theta)^2 + (2a - \sin\theta)^2 \leqslant 4 \\ &\Leftrightarrow 2a(\cos\theta - 2\sin\theta) \leqslant 3 - 5a^2 \\ &\Leftrightarrow -2\sqrt{5}a\sin(\theta - \varphi) \leqslant 3 - 5a^2 \\ &\Rightarrow 2\sqrt{5}\,|a| \leqslant 3 - 5a^2 \\ &\Rightarrow |a| \leqslant \frac{\sqrt{5}}{5}. \end{aligned}$$

So the range of a is $\left[-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right]$.

9 Suppose points F_1, F_2 are the left and right foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ respectively, and point P is on line l: $x-\sqrt{3}y+8+2\sqrt{3}=0$. When $\angle F_1PF_2$ reaches the maximum, the value of ratio $\frac{|PF_1|}{|PF_2|}$ is _______.

Solution Euclidean geometry tells us that, $\angle F_1PF_2$ reaches the maximum only if the circle through points F_1, F_2, P is tangent to the line l at P. Now suppose l intercepts the x-axis at point $A(-8-2\sqrt{3}, 0)$. Then $\angle APF_1=\angle AF_2P$, and that means $\triangle APF_1 \backsim \triangle AF_2P$. So

$$\frac{|PF_1|}{|PF_2|}=\frac{|AP|}{|AF_2|}.$$

By using the power of points theorem, we have

$$|AP|^2=|AF_1|\cdot|AF_2|.$$

As $F_1(-2\sqrt{3}, 0)$, $F_2(2\sqrt{3}, 0)$, $A(-8-2\sqrt{3}, 0)$, so

$$|AF_1|=8,\ |AF_2|=8+4\sqrt{3}.$$

Then we get

$$\frac{|PF_1|}{|PF_2|}=\sqrt{\frac{|AF_1|}{|AF_2|}}=\sqrt{\frac{8}{8+4\sqrt{3}}}$$
$$=\sqrt{4-2\sqrt{3}}=\sqrt{3}-1.$$

10 Suppose four solid iron balls are placed in a cylinder with the radius of 1 cm, such that every two of the four balls are tangent to each other, and the two balls in the lower layer are tangent to the cylinder base. Now put water into the cylinder. Then, to just submerge all the balls, we need a volume of ________ cm^3 water.

Solution Let points O_1, O_2, O_3, O_4 be the centers of the four solid iron balls respectively, with O_1, O_2 belonging to the two balls in the lower layer, and A, B, C, D be the projective points of O_1, O_2, O_3, O_4 on the base of the cylinder. $ABCD$ constitute a square with the side of $\frac{\sqrt{2}}{2}$. So the height of the water in the cylinder must be $1+\frac{\sqrt{2}}{2}$, so that all the balls are just immersed. Hence the volume of water we need is

$$\pi\left(1+\frac{\sqrt{2}}{2}\right)-4\times\frac{4}{3}\pi\left(\frac{1}{2}\right)^3=\left(\frac{1}{3}+\frac{\sqrt{2}}{2}\right)\pi.$$

11 The number of real solutions for equation

$$(x^{2006}+1)(1+x^2+x^4+\cdots+x^{2004})=2006x^{2005}$$

is ________.

Solution We have

$$\begin{aligned}&(x^{2006}+1)(1+x^2+x^4+\cdots+x^{2004})=2006x^{2005}\\ \Leftrightarrow&\left(x+\frac{1}{x^{2005}}\right)(1+x^2+x^4+\cdots+x^{2004})=2006\\ \Leftrightarrow&\ x+x^3+x^5+\cdots+x^{2005}+\frac{1}{x^{2005}}+\frac{1}{x^{2003}}+\frac{1}{x^{2001}}\\ &+\cdots+\frac{1}{x}=2006\\ \Leftrightarrow&\ 2006=x+\frac{1}{x}+x^3+\frac{1}{x^3}+\cdots+x^{2005}+\frac{1}{x^{2005}}\\ &\qquad\geqslant 2\times1003=2006,\end{aligned}$$

where the equal holds if and only if $x=\frac{1}{x}$, $x^3=\frac{1}{x^3}$, $\cdots$, $x^{2005}=\frac{1}{x^{2005}}$. Then $x=\pm1$.

Since $x\leqslant 0$ does not satisfy the original equation, $x=1$ is the only solution. So the number of real solution is 1.

12 Suppose there are 8 white balls and 2 red balls in a packet. Each time one ball is drawn and replaced by a white one. Then the probability of drawing out all of the red balls just in the fourth draw is ______.

Solution The following three cases can satisfy the condition.

	1st draw	2nd draw	3rd draw	4th draw
Case 1	Red	White	White	Red
Case 2	White	Red	White	Red
Case 3	White	White	Red	Red

So the probability

$$\begin{aligned} P &= P(\text{Case } 1) + P(\text{Case } 2) + P(\text{Case } 3) \\ &= \frac{2}{10} \times \left(\frac{9}{10}\right)^2 \times \frac{1}{10} + \frac{8}{10} \times \frac{2}{10} \times \frac{9}{10} \times \frac{1}{10} \\ &\quad + \left(\frac{8}{10}\right)^2 \times \frac{2}{10} \times \frac{1}{10} \\ &= 0.0434. \end{aligned}$$

Part III Word Problems (Questions 13 to 15, each carries 20 marks.)

13 Given an integer $n \geqslant 2$, define $M_0(x_0, y_0)$ to be an intersection point of the parabola $y^2 = nx - 1$ and the line $y = x$. Prove that for any positive integer m, there exists an integer $k \geqslant 2$ such that (x_0^m, y_0^m) is an intersection point of $y^2 = kx - 1$ and $y = x$.

Proof Since $M_0(x_0, y_0)$ is an intersection point of $y^2 = nx -$

1 and $y = x$, we get $x_0 = y_0 = \frac{n \pm \sqrt{n^2 - 4}}{2}$. Then obviously $x_0 + \frac{1}{x_0} = n$.

Let (x_0^m, y_0^m) be an intersection point of $y^2 = kx - 1$ and $y = x$. Then we get

$$k = x_0^m + \frac{1}{x_0^m}.$$

We denote $k_m = x_0^m + \frac{1}{x_0^m}$. Then,

$$k_{m+1} = k_m\left(x_0 + \frac{1}{x_0}\right) - k_{m-1} = nk_m - k_{m-1} (m \geqslant 2). \qquad ①$$

Since $k_1 = n$ is an integer,

$$k_2 = x_0^2 + \frac{1}{x_0^2} = \left(x_0 + \frac{1}{x_0}\right)^2 - 2 = n^2 - 2$$

is also an integer. Then, by the principle of mathematical induction and ①, we conclude that for any positive integer m, $k_m = x_0^m + \frac{1}{x_0^m}$ is a positive integer too. Let $k = x_0^m + \frac{1}{x_0^m}$. So (x_0^m, y_0^m) is an intersection point of $y^2 = kx - 1$ and $y = x$.

14 Let 2006 be expressed as the sum of five positive integers x_1, x_2, x_3, x_4, x_5, and $S = \sum_{1 \leqslant i < j \leqslant 5} x_i x_j$. We ask:

(1) What value of x_1, x_2, x_3, x_4, x_5 will make S the maximum?

(2) Further, if $|x_i - x_j| \leqslant 2$ for any $1 \leqslant i, j \leqslant 5$, then what value of x_1, x_2, x_3, x_4, x_5 will make S the minimum? You should prove your answer.

Solution (1) Obviously the number of the values of S is finite, so the maximum and minimum exist. Suppose $x_1 + x_2 +$

$x_3 + x_4 + x_5 = 2006$ such that $S = \sum_{1 \leqslant i < j \leqslant 5} x_i x_j$ reaches the maximum, we must have

$$|x_i - x_j| \leqslant 1, \ (1 \leqslant i, j \leqslant 5). \qquad ①$$

Otherwise, assume that ① does not hold. Without loss of generality, suppose $x_1 - x_2 \geqslant 2$. Let $x_1' = x_1 - 1$, $x_2' = x_2 + 1$, $x_i' = x_i (i = 3, 4, 5)$. We have

$$\begin{aligned} x_1' + x_2' + x_3' + x_4' + x_5' &= x_1 + x_2 + x_3 + x_4 + x_5 \\ &= 2006, \end{aligned}$$

$$\begin{aligned} S &= \sum_{1 \leqslant i < j \leqslant 5} x_i x_j \\ &= x_1 x_2 + (x_1 + x_2)(x_3 + x_4 + x_5) + x_3 x_4 + x_3 x_5 + x_4 x_5, \\ S' &= x_1' x_2' + (x_1' + x_2')(x_3 + x_4 + x_5) \\ &\quad + x_3 x_4 + x_3 x_5 + x_4 x_5. \end{aligned}$$

So

$$S' - S = x_1' x_2' - x_1 x_2 > 0.$$

This contradicts the assumption that S is the maximum.

Therefore $|x_i - x_j| \leqslant 1$ $(1 \leqslant i, j \leqslant 5)$. And it is easy to check that S reaches the maximum when

$$x_1 = 402, \ x_2 = x_3 = x_4 = x_5 = 401.$$

(2) If we neglect the order in x_1, x_2, x_3, x_4, x_5, there could only be three cases:

(a) 402, 402, 402, 400, 400;

(b) 402, 402, 401, 401, 400;

(c) 402, 401, 401, 401, 401.

That satisfy $x_1 + x_2 + x_3 + x_4 + x_5 = 2006$ and $|x_i - x_j| \leqslant 2$.

Cases (b) and (c) can be obtained from Case (a) by setting $x_i' = x_i - 1$, $x_j' = x_j + 1$. What we have done in (1) tells us that each step like this will make $S' = \sum_{1\leqslant i<j\leqslant 5} x'_i x'_j$ greater. So S is the minimum in Case (a), i.e. $x_1 = x_2 = x_3 = 402$, $x_4 = x_5 = 400$.

15 Suppose $f(x) = x^2 + a$. Define $f^1(x) = f(x)$, $f^n(x) = f(f^{n-1}(x))$, $n = 2, 3, \cdots$, and let $M = \{a \in \mathbf{R} \mid | f^n(0) | \leqslant 2$, for any $n \in \mathbf{N}\}$. Prove that $M = \left[-2, \frac{1}{4}\right]$.

Proof (a) When $a < -2$, then $| f^1(0) | = | a | > 2$, therefore $a \notin M$.

(b) When $-2 \leqslant a < 0$, we have $| f^1(0) | = | a | \leqslant 2$. Assume that $| f^{k-1}(0) | \leqslant | a | \leqslant 2$ for $k \geqslant 2$. Since $a^2 \leqslant -2a$ for $-2 \leqslant a < 0$, we get that

$$\begin{aligned} -2 \leqslant -| a | = a \leqslant f^k(0) &= (f^{k-1}(0))^2 + a \\ &\leqslant a^2 + a \leqslant | a | \leqslant 2. \end{aligned}$$

By the principle of mathematical induction, we conclude that $| f^n(0) | \leqslant a \leqslant 2$ ($\forall n \geqslant 1$).

(c) When $0 \leqslant a \leqslant \frac{1}{4}$, we have $| f^1(0) | = | a | \leqslant \frac{1}{2}$. Assume that $| f^{k-1}(0) | \leqslant \frac{1}{2}$ for $k \geqslant 2$. We get

$$| f^k(0) | \leqslant | f^{k-1}(0) |^2 + a \leqslant \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}.$$

By the principle of mathematical induction, we conclude that $| f^n(0) | \leqslant \frac{1}{2}$ ($\forall n \geqslant 1$).

From (b) and (c), we obtain $\left[-2, \frac{1}{4}\right] \subseteq M$.

(d) When $a > \frac{1}{4}$, define $a_n = f^n(0)$. We have

$$a_{n+1} = f^{n+1}(0) = f(f^n(0)) = f(a_n) = a_n^2 + a,$$

then $a_n > a > \frac{1}{4}$ for any $n \geqslant 1$. Since

$$a_{n+1} - a_n = a_n^2 - a_n + a = \left(a_n - \frac{1}{2}\right)^2 + a - \frac{1}{4} \geqslant a - \frac{1}{4},$$

we get

$$\begin{aligned} a_{n+1} - a &= a_{n+1} - a_1 \\ &= (a_{n+1} - a_n) + \cdots + (a_2 - a_1) \\ &\geqslant n\left(a - \frac{1}{4}\right). \end{aligned}$$

Therefore, when $n > \dfrac{2-a}{a - \frac{1}{4}}$, we have

$$a_{n+1} \geqslant n\left(a - \frac{1}{4}\right) + a > 2 - a + a = 2.$$

And that means $a \notin M$.

From (a)—(d), we proved that $M = \left[-2, \frac{1}{4}\right]$.

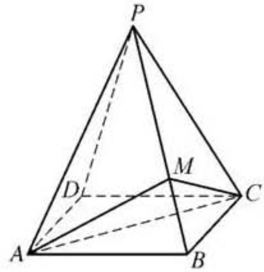
2007 (Tianjin)

Popularization Committee of Chinese Mathematical Society and Tianjin Mathematical Society were responsible for the assignment of competition problems in the first round and the extra round of the competition.

Part I Multiple-Choice Questions (Questions 1 to 6, each carries 6 marks.)

1 Given a right square pyramid P-$ABCD$ with $\angle APC =$

60°, as shown in the figure, prove that the cosine of the plane angle of the dihedral angle A-PB-C is ( ).



(A) $\frac{1}{7}$ (B) $-\frac{1}{7}$

(C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Solution On PAB, draw $AM \perp PB$ with M as the foot drop, connecting CM and AC, as seen in the figure. Then $\angle AMC$ is the plane angle of the dihedral angle $A-PB-C$. We may assume that $AB = 2$. Then we get $PA = AC = 2\sqrt{2}$, and the vertical height of $\triangle PAB$ with AB as base is $\sqrt{7}$. So we have $2 \times \sqrt{7} = AM \cdot 2\sqrt{2}$. That means $AM = \sqrt{\frac{7}{2}} = CM$. By the Cosine Rule we have

$$\cos \angle AMC = \frac{AM^2 + CM^2 - AC^2}{2 \cdot AM \cdot CM} = -\frac{1}{7}.$$

Answer: B.

2 Suppose real number a satisfies $|2x - a| + |3x - 2a| \geqslant a^2$ for any $x \in \mathbf{R}$. Then a lies exactly in ( ).

(A) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ (B) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(C) $\left[-\frac{1}{4}, \frac{1}{3}\right]$ (D) $[-3, 3]$

Solution Let $x = \frac{2}{3}a$. Then we have $|a| \leqslant \frac{1}{3}$. Therefore (B) and (D) are excluded. By symmetry, (C) is also excluded. Then only (A) can be correct.

In general, for any $k \in \mathbf{R}$, let $x = \frac{1}{2}ka$. Then the original inequality becomes

$$|a| \cdot |k-1| + \frac{3}{2}|a| \cdot \left|k - \frac{4}{3}\right| \geqslant |a|^2.$$

This is equivalent to

$$|a| \leqslant |k-1| + \frac{3}{2}\left|k - \frac{4}{3}\right|.$$

We have

$$|k-1| + \frac{3}{2}\left|k-\frac{4}{3}\right| = \begin{cases} \frac{5}{2}k - 3, & k \geqslant \frac{4}{3}, \\ 1 - \frac{1}{2}k, & 1 \leqslant k < \frac{4}{3}, \\ 3 - \frac{5}{2}k, & k < 1. \end{cases}$$

So

$$\min_{k \in \mathbf{R}}\left\{|k-1| + \frac{3}{2}\left|k-\frac{4}{3}\right|\right\} = \frac{1}{3}.$$

The inequality is reduced to $|a| \leqslant \frac{1}{3}$. Answer: A.

3 Nine balls of the same size and color, numbered 1, 2, $\cdots$, 9, were put into a packet. Now A draws a ball from the packet, noted that it is of number a, and puts back it. Then B also draws a ball from the packet and noted that it is of number b. Then the probability for the inequality $a - 2b + 10 > 0$ to hold is ( ).

(A) $\frac{52}{81}$ (B) $\frac{59}{81}$ (C) $\frac{60}{81}$ (D) $\frac{61}{81}$

Solution Since each has equally 9 different possible results for

A and B to draw a ball from the packet independently, the total number of possible events is $9^2 = 81$. From $a - 2b + 10 > 0$ we get $2b < a + 10$. We find that when $b = 1, 2, 3, 4, 5$, a can take any value in $1, 2, \cdots, 9$ to make the inequality hold. Then we have $9 \times 5 = 45$ admissible events.

When $b = 6$, a can be $3, 4, \cdots, 9$, and there are 7 admissible events.

When $b = 7$, a can be $5, 6, 7, 8, 9$, and there are 5 admissible events.

When $b = 8$, a can be $7, 8, 9$, and there are 3 admissible events.

When $b = 9$, a can only be 9, and there is 1 admissible events.

So the required probability is $\frac{45 + 7 + 5 + 3 + 1}{81} = \frac{61}{81}$.

Answer: D.

4 Let $f(x) = 3\sin x + 2\cos x + 1$. If real numbers a, b, c are such that $af(x) + bf(x - c) = 1$ holds for any $x \in \mathbf{R}$, then $\frac{b\cos c}{a}$ equals ( ).

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) -1 (D) 1

Solution Let $c = \pi$. Then $f(x) + f(x - c) = 2$ for any $x \in \mathbf{R}$. Now let $a = b = \frac{1}{2}$, and $c = \pi$. We have

$$af(x) + bf(x - c) = 1$$

for any $x \in \mathbf{R}$. Consequently, $\frac{b\cos c}{a} = -1$. So Answer is (C).

More generally, we have

$$f(x) = \sqrt{13}\sin(x + \varphi) + 1,$$
$$f(x - c) = \sqrt{13}\sin(x + \varphi - c) + 1,$$

where $0 < \varphi < \frac{\pi}{2}$ and $\tan\varphi = \frac{2}{3}$. Then $af(x) + bf(x - c) = 1$ becomes

$$\sqrt{13}a\sin(x + \varphi) + \sqrt{13}b\sin(x + \varphi - c) + a + b = 1.$$

That is,

$$\sqrt{13}a\sin(x + \varphi) + \sqrt{13}b\sin(x + \varphi)\cos c$$
$$- \sqrt{13}b\sin c\cos(x + \varphi) + (a + b - 1) = 0.$$

Therefore

$$\sqrt{13}(a + b\cos c)\sin(x + \varphi) - \sqrt{13}b\sin c\cos(x + \varphi)$$
$$+ (a + b - 1) = 0.$$

Since the equality above holds for any $x \in \mathbf{R}$, we must have

$$\begin{cases} a + b\cos c = 0, & ① \\ b\sin c = 0, & ② \\ a + b - 1 = 0. & ③ \end{cases}$$

If $b = 0$, then $a = 0$ from ①, and this contradicts ③. So $b \neq 0$, and $\sin c = 0$ from ②. Therefore $c = 2k\pi + \pi$ or $c = 2k\pi$ $(k \in \mathbf{Z})$.

If $c = 2k\pi$, then $\cos c = 1$, and it leads to a contradiction between ① and ③. So $c = 2k\pi + \pi$ $(k \in \mathbf{Z})$ and $\cos c = -1$. From ① and ③, we get $a = b = \frac{1}{2}$. Consequently, $\frac{b\cos c}{a} = -1$.

5 Given two fixed circles with O_1 and O_2 as their center respectively, a circle P moves in a way such that it is tangent to both of them. Then the locus of the center of P cannot be ( ).

Solution Suppose the radii of the two fixed circles are r_1 and r_2 respectively, and $|O_1O_2| = 2c$. Then, in general, the locus of the center of P is given by two conic curves with O_1, O_2 as the foci, and $\frac{2c}{r_1+r_2}$, $\frac{2c}{|r_1-r_2|}$ the eccentricities, respectively. (When $r_1 = r_2$, the perpendicular bisector of O_1O_2 is a part of the locus. When $c = 0$, the locus is given by two concentric circles.)

When $r_1 = r_2$ and $r_1 + r_2 < 2c$, the locus of the center of P is like (B). When $0 < 2c < |r_1 - r_2|$, the locus is like (C). When $r_1 \neq r_2$ and $r_1 + r_2 < 2c$, the locus is like (D). Since the foci of the ellipse and the hyperbola in (A) are not identical, the locus of the center of P cannot be (A). Answer: A.

6 Let A and B be two subsets of $\{1, 2, 3, \cdots, 100\}$, satisfying $|A| = |B|$ and $A \cap B = \varnothing$. If $n \in A$ always implies $2n + 2 \in B$, then the maximum of $|A \cup B|$ is ( ).

(A) 62 (B) 66 (C) 68 (D) 74

Solution We will first prove that $|A \cup B| \leqslant 66$, or equivalently $|A| \leqslant 33$. For this purpose, we only need to prove that, if A is a subset of $\{1, 2, \cdots, 49\}$ with 34 elements, then there must exist $n \in A$ such that $2n+2 \in A$. The proof is as follows.

Divide $\{1, 2, \cdots, 49\}$ into 33 subsets:

$\{1, 4\}$, $\{3, 8\}$, $\{5, 12\}$, $\cdots$, $\{23, 48\}$, 12 subsets;

$\{2, 6\}$, $\{10, 22\}$, $\{14, 30\}$, $\{18, 38\}$, 4 subsets;

$\{25\}$, $\{27\}$, $\{29\}$, $\cdots$, $\{49\}$, 13 subsets;

$\{26\}$, $\{34\}$, $\{42\}$, $\{46\}$, 4 subsets.

By the Pigeonhole Principle we know that there exists at least one subset with 2 elements among them which is also a subset of A. That means there exists $n \in A$ such that $2n + 2 \in A$.

On the other hand, let

$A = \{1, 3, 5, \cdots, 23, 2, 10, 14, 18, 25, 27, 29, \cdots, 49, 26, 34, 42, 46\}$

$B = \{2n+2 \mid n \in A\}$. We find that A and B satisfy the condition and $|A \cup B| = 66$. Answer: B.

Part II Short-Answer Questions (Questions 7 to 12, each carries 9 marks.)

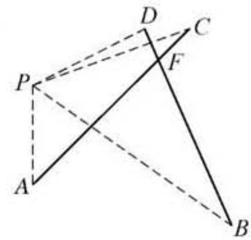
7 Given four fixed points $A(-3, 0)$, $B(1, -1)$, $C(0, 3)$, $D(-1, 3)$ and a variable point P in a plane rectangular coordinates system, the minimum of $|PA|+|PB|+|PC|+|PD|$ is ________.

Solution As shown in the figure, assuming that AC and BD meet at point F, we have

$$|PA|+|PC| \geqslant |AC| = |FA|+|FC|$$

and

$|PB|+|PD| \geqslant |BD| = |FB|+|FD|.$



When P coincides with F, $|PA|+|PB|+|PC|+|PD|$ reaches the minimum. That is $|AC|+|BD| = 3\sqrt{2}+2\sqrt{5}$. So $3\sqrt{2}+2\sqrt{5}$ is the required answer.

8 Given $\triangle ABC$ and $\triangle AEF$ such that B is the midpoint of EF. Also, $AB = EF = 1$, $BC = 6$, $CA = \sqrt{33}$, and $\overrightarrow{AB} \cdot \overrightarrow{AE} + \overrightarrow{AC} \cdot \overrightarrow{AF} = 2$. The cosine of the angle between $\overrightarrow{EF}$ and $\overrightarrow{BC}$ is ________.

Solution We have

$$\begin{aligned} 2 &= \overrightarrow{AB} \cdot \overrightarrow{AE} + \overrightarrow{AC} \cdot \overrightarrow{AF} \\ &= \overrightarrow{AB} \cdot (\overrightarrow{AB} + \overrightarrow{BE}) + \overrightarrow{AC} \cdot (\overrightarrow{AB} + \overrightarrow{BF}), \end{aligned}$$

i.e.

$$\overrightarrow{AB}^2 + \overrightarrow{AB} \cdot \overrightarrow{BE} + \overrightarrow{AC} \cdot \overrightarrow{AB} + \overrightarrow{AC} \cdot \overrightarrow{BF} = 2.$$

As $\overrightarrow{AB}^2 = 1$,

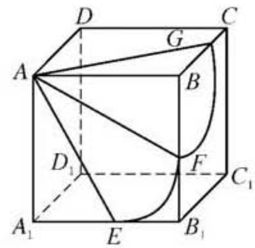
$$\overrightarrow{AC} \cdot \overrightarrow{AB} = \sqrt{33} \times 1 \times \frac{33+1-36}{2 \times \sqrt{33} \times 1} = -1$$

and $\overrightarrow{BE} = -\overrightarrow{BF}$, we get

$$1 + \overrightarrow{BF} \cdot (\overrightarrow{AC} - \overrightarrow{AB}) - 1 = 2,$$

i.e. $\overrightarrow{BF} \cdot \overrightarrow{BC} = 2$. Defining θ as the angle between $\overrightarrow{EF}$ and $\overrightarrow{BC}$, we get $|\overrightarrow{BF}| \cdot |\overrightarrow{BC}| \cdot \cos\theta = 2$ or $3\cos\theta = 2$. So $\cos\theta = \dfrac{2}{3}$.

9 Given a unit cube $ABCD\text{-}A_1B_1C_1D_1$, construct a ball with point A as the center and of radius $\dfrac{2\sqrt{3}}{3}$. Then the length of the curves resulting from the intersection between the surfaces of the ball and cube is ________.



Solution As shown in the figure, the surface of the ball intersects all of the six surfaces of the cube. The intersection curves are divided into two kinds: One kind lies on the three surfaces including vertex A respectively, that is AA_1B_1B, $ABCD$, and AA_1D_1D; while the other lies on the three surfaces not including A, that is CC_1D_1D, $A_1B_1C_1D_1$ and BB_1C_1C.

On surface AA_1B_1B, the intersection curve is arc $\overset{\frown}{EF}$ which lies on a circle with A as the center. Since $AE=\frac{2\sqrt{3}}{3}$, $AA_1=1$, so $\angle A_1AE=\frac{\pi}{6}$. In the same way $\angle BAF=\frac{\pi}{6}$. Therefore $\angle EAF=\frac{\pi}{6}$. That means the length of arc $\overset{\frown}{EF}$ is $\frac{2\sqrt{3}}{3}\cdot\frac{\pi}{6}=\frac{\sqrt{3}}{9}\pi$. There are three arcs of this category.

On surface BB_1C_1C, the intersection curve is arc $\overset{\frown}{FG}$ which lies on a circle centred at B. The radius equals $\frac{\sqrt{3}}{3}$ and $\angle FBG=\frac{\pi}{2}$. So the length of $\overset{\frown}{FG}$ is $\frac{\sqrt{3}}{3}\cdot\frac{\pi}{2}=\frac{\sqrt{3}}{6}\pi$. There are also three arcs of this category.

In summary, the total length of all intersection curves is

$$3\times\frac{\sqrt{3}}{9}\pi+3\times\frac{\sqrt{3}}{6}\pi=\frac{5\sqrt{3}\pi}{6}.$$

10 Let $\{a_n\}$ be an arithmetic progression with common difference $d\,(d \neq 0)$ and $\{b_n\}$ be a geometric progression with common ratio q, where q is a positive rational number less than 1. If $a_1 = d$, $b_1 = d^2$ and $\frac{a_1^2 + a_2^2 + a_3^2}{b_1 + b_2 + b_3}$ is a positive integer, then q equals _______.

Solution As

$$\frac{a_1^2 + a_2^2 + a_3^2}{b_1 + b_2 + b_3} = \frac{a_1^2 + (a_1 + d)^2 + (a_1 + 2d)^2}{b_1 + b_1 q + b_1 q^2}$$
$$= \frac{14}{1 + q + q^2} = m$$

is a positive integer, we get $1 + q + q^2 = \frac{14}{m}$. Then

$$q = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{14}{m} - 1} = -\frac{1}{2} + \sqrt{\frac{56 - 3m}{4m}}.$$

Since q is a positive rational number less than 1, we have $1 < \frac{14}{m} < 3$, i.e. $5 \leqslant m \leqslant 13$, and $\frac{56 - 3m}{4m}$ is the square of a rational number. We can verify that only $m = 8$ meets the required. That means $q = \frac{1}{2}$.

11 Given $f(x) = \frac{\sin(\pi x) - \cos(\pi x) + 2}{\sqrt{x}}$ for $\frac{1}{4} \leqslant x \leqslant \frac{5}{4}$, the minimum of $f(x)$ is _______.

Solution By rewritting $f(x)$, we have

$$f(x) = \frac{\sqrt{2}\sin\left(\pi x - \frac{\pi}{4}\right) + 2}{\sqrt{x}}$$

for $\frac{1}{4} \leqslant x \leqslant \frac{5}{4}$. Define $g(x) = \sqrt{2}\sin\left(\pi x - \frac{\pi}{4}\right)$, where

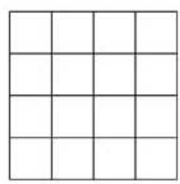
$\frac{1}{4} \leqslant x \leqslant \frac{5}{4}$. Then $g(x) \geqslant 0$, and $g(x)$ is monotone increasing on $\left[\frac{1}{4}, \frac{3}{4}\right]$, and monotone decreasing on $\left[\frac{3}{4}, \frac{5}{4}\right]$.

Further, the graph of $y = g(x)$ is symmetric about $x = \frac{3}{4}$, i.e. for any $x_1 \in \left[\frac{1}{4}, \frac{3}{4}\right]$ there exists $x_2 \in \left[\frac{3}{4}, \frac{5}{4}\right]$ such that $g(x_2) = g(x_1)$. Then

$$f(x_1) = \frac{g(x_1) + 2}{\sqrt{x_1}} = \frac{g(x_2) + 2}{\sqrt{x_1}} \geqslant \frac{g(x_2) + 2}{\sqrt{x_2}} = f(x_2).$$

On the other hand, $f(x)$ is monotone decreasing on $\left[\frac{3}{4}, \frac{5}{4}\right]$. Therefore $f(x) \geqslant f\left(\frac{5}{4}\right) = \frac{4\sqrt{5}}{5}$. That means the minimum value of $f(x)$ on $\left[\frac{1}{4}, \frac{5}{4}\right]$ is $\frac{4\sqrt{5}}{5}$.

12 Four letters, two "a"s and two "b"s, are filled into 16 cells of a matrix as shown in figure. It is required that each cell contains at most one letter, and each row or column cannot contain the same letters. Then there are ________ different ways that the matrix can be filled. (A numerical answer is needed.)



Solution It is easy to see that there are $\binom{4}{2} \cdot P_2^4 = 72$ different ways to put two "a"s into the matrix such that each row and each column contains at most one "a". Similarly, there are also $\binom{4}{2} \cdot P_2^4 = 72$ different ways for two "b"s to do the same thing. By the multiplicative principle we get 72^2 ways. Among them we exclude 72 cases in which two "b"s occupy the same cells as the two "a"s do, and exclude $\binom{16}{1} \cdot P_2^9 = 16 \times 72$ cases in which one "a" shares the same cell with a "b". Therefore there are $72^2 - 72 - 16 \times 72 = 3960$ different ways that meet the requirement.

Part III Word Problems (Questions 13 to 15, each carries 20 marks.)

13 Let $a_n = \sum_{k=1}^{n} \frac{1}{k(n+1-k)}$. Prove that $a_{n+1} < a_n$ for $n \geqslant 2$.

Proof As

$$\frac{1}{k(n+1-k)} = \frac{1}{n+1}\left(\frac{1}{k} + \frac{1}{n+1-k}\right),$$

we get $a_n = \frac{2}{n+1}\sum_{k=1}^{n} \frac{1}{k}$. Then for $n \geqslant 2$ we have

$$\begin{aligned}
\frac{1}{2}(a_n - a_{n+1}) &= \frac{1}{n+1}\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{n+2}\sum_{k=1}^{n+1} \frac{1}{k} \\
&= \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{(n+1)(n+2)} \\
&= \frac{1}{(n+1)(n+2)}\left(\sum_{k=1}^{n} \frac{1}{k} - 1\right) \\
&> 0.
\end{aligned}$$

That means $a_{n+1} < a_n$.

14 Suppose line l through point $(0, 1)$ and curve $C: y = x + \frac{1}{x}(x > 0)$ intersect at two different points M and N. Find the locus of the intersection points of two tangent lines of curve C at M and N respectively.

Solution Denote the coordinates of M and N as (x_1, y_1) and (x_2, y_2) respectively. Denote the tangent lines of C at M and N by l_1 and l_2 respectively, with their intersection point being $P(x_p, y_p)$. Suppose the slope ratio of line l is k. Then we can write the equation of l as $y = kx + 1$.

Eliminating y from

$$\begin{cases} y = x + \dfrac{1}{x}, \\ y = kx + 1, \end{cases}$$

we get $x + \dfrac{1}{x} = kx + 1$, i.e. $(k-1)x^2 + x - 1 = 0$. By the assumption, we know that the equation has two distinctive real roots, x_1 and x_2, on $(0, +\infty)$. Then $k \neq 1$, and

$$\Delta = 1 + 4(k-1) > 0, \qquad ①$$

$$x_1 + x_2 = \frac{1}{1-k} > 0, \qquad ②$$

$$x_1 x_2 = \frac{1}{1-k} > 0. \qquad ③$$

From the above we get $\dfrac{3}{4} < k < 1$.

We find the derivative of $y = x + \dfrac{1}{x}$ as $y' = 1 - \dfrac{1}{x^2}$. Then $y'|_{x=x_1} = 1 - \dfrac{1}{x_1^2}$ and $y'|_{x=x_2} = 1 - \dfrac{1}{x_2^2}$. Therefore, the equation of line l_1 is

$$y - y_1 = \left(1 - \frac{1}{x_1^2}\right)(x - x_1)$$

or

$$y - \left(x_1 + \frac{1}{x_1}\right) = \left(1 - \frac{1}{x_1^2}\right)(x - x_1).$$

After simplification, we get

$$y = \left(1 - \frac{1}{x_1^2}\right)x + \frac{2}{x_1}. \qquad ④$$

In the same way, we get the equation of l_2,

$$y = \left(1 - \frac{1}{x_2^2}\right)x + \frac{2}{x_2}. \qquad ⑤$$

By ④ − ⑤, we get

$$\left(\frac{1}{x_2^2} - \frac{1}{x_1^2}\right)x_p + \frac{2}{x_1} - \frac{2}{x_2} = 0.$$

Since $x_1 \neq x_2$, we have

$$x_p = \frac{2x_1x_2}{x_1 + x_2}. \qquad ⑥$$

Substituting ② and ③ into ⑥, we obtain $x_p = 2$.

By ④ + ⑤, we obtain

$$2y_p = \left(2 - \left(\frac{1}{x_1^2} + \frac{1}{x_2^2}\right)\right)x_p + 2\left(\frac{1}{x_1} + \frac{1}{x_2}\right), \qquad ⑦$$

where

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1x_2} = 1,$$

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} = \frac{x_1^2 + x_2^2}{x_1^2\, x_2^2} = \frac{(x_1 + x_2)^2 - 2x_1x_2}{x_1^2\, x_2^2}$$

$$= \left(\frac{x_1 + x_2}{x_1x_2}\right)^2 - \frac{2}{x_1x_2} = 2k - 1.$$

Substituting it into ⑦, we have $2y_p = (3 - 2k)x_p + 2$. Since $x_p = 2$, then $y_p = 4 - 2k$. As $\frac{3}{4} < k < 1$, we get $2 < y_p < \frac{5}{2}$. Therefore, the locus of point P is the segment between $(2, 2)$ and $(2, 2.5)$ (not including the endpoints).

15 Suppose $f(x+2\pi)=f(x)$ for any $x\in\mathbf{R}$. Prove: there are $f_i(x)(i=1, 2, 3, 4)$ such that

(1) $f_i(x)$ $(i=1, 2, 3, 4)$ is an even function, and $f_i(x+\pi)=f_i(x)$ for any $x\in\mathbf{R}$;

(2) $f(x)=f_1(x)+f_2(x)\cos x+f_3(x)\sin x+f_4(x)\sin 2x$ for any $x\in\mathbf{R}$.

Proof Let $g(x)=\dfrac{f(x)+f(-x)}{2}$, and $h(x)=\dfrac{f(x)-f(-x)}{2}$. Then $f(x)=g(x)+h(x)$, $g(x)$ is an even function, $h(x)$ is an odd function, and $g(x+2\pi)=g(x)$, $h(x+2\pi)=h(x)$ for any $x\in\mathbf{R}$.

Define

$$f_1(x)=\frac{g(x)+g(x+\pi)}{2},$$

$$f_2(x)=\begin{cases}\dfrac{g(x)-g(x+\pi)}{2\cos x}, & x\neq k\pi+\dfrac{\pi}{2},\\[2mm] 0, & x=k\pi+\dfrac{\pi}{2},\end{cases}$$

$$f_3(x)=\begin{cases}\dfrac{h(x)-h(x+\pi)}{2\sin x}, & x\neq k\pi,\\[2mm] 0, & x=k\pi,\end{cases}$$

$$f_4(x)=\begin{cases}\dfrac{h(x)+h(x+\pi)}{2\sin 2x}, & x\neq \dfrac{k\pi}{2},\\[2mm] 0, & x=\dfrac{k\pi}{2},\end{cases}$$

where k is an arbitrary integer. It is easy to check that $f_i(x)$ $(i=1, 2, 3, 4)$ satisfy (1).

Next we prove that $f_1(x)+f_2(x)\cos x=g(x)$ for any $x\in\mathbf{R}$. When $x\neq k\pi+\dfrac{\pi}{2}$, it is obviously true. When $x=k\pi+\dfrac{\pi}{2}$, we have

$$f_1(x)+f_2(x)\cos x=f_1(x)=\frac{g(x)+g(x+\pi)}{2},$$

and

$$\begin{aligned}g(x+\pi)&=g\left(k\pi+\frac{3\pi}{2}\right)=g\left(k\pi+\frac{3\pi}{2}-2(k+1)\pi\right)\\&=g\left(-k\pi-\frac{\pi}{2}\right)=g\left(k\pi+\frac{\pi}{2}\right)=g(x).\end{aligned}$$

The proof is complete.

Further we prove that $f_3(x)\sin x+f_4(x)\sin 2x=h(x)$ for any $x\in\mathbf{R}$. When $x\neq\frac{k\pi}{2}$, it is obviously true. When $x=k\pi$, we have

$$h(x)=h(k\pi)=h(k\pi-2k\pi)=h(-k\pi)=-h(k\pi).$$

That means $h(x)=h(k\pi)=0$. In this case,

$$f_3(x)\sin x+f_4(x)\sin 2x=0.$$

Therefore $h(x)=f_3(x)\sin x+f_4(x)\sin 2x$.

When $x=k\pi+\frac{\pi}{2}$, we have

$$\begin{aligned}h(x+\pi)&=h\left(k\pi+\frac{3\pi}{2}\right)=h\left(k\pi+\frac{3\pi}{2}-2(k+1)\pi\right)\\&=h\left(-k\pi-\frac{\pi}{2}\right)=-h\left(k\pi+\frac{\pi}{2}\right)=-h(x).\end{aligned}$$

So

$$f_3(x)\sin x=\frac{h(x)-h(x+\pi)}{2}=h(x).$$

Furthermore, $f_4(x)\sin 2x=0$. Therefore

$$h(x)=f_3(x)\sin x+f_4(x)\sin 2x.$$

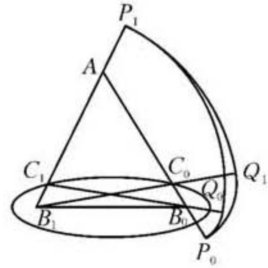
This completes the proof.

In conclusion, $f_i(x)$ $(i=1, 2, 3, 4)$ satisfy (2).

China Mathematical Competition (Extra Test)

2006 (Zhejiang)

1 Suppose an ellipse with points B_0 and B_1 as the foci intercepts side AB_i of $\triangle AB_0B_1$ at C_i $(i = 0, 1)$. Taking an arbitrary point P_0 on the extending line of AB_0, draw arc $\widehat{P_0Q_0}$ with B_0, B_0P_0 as the center and radius respectively, intercepting the extending line of C_1B_0 at Q_0. Draw arc $\widehat{Q_0P_1}$ with C_1, C_1Q_0 as the center and radius respectively,



intercepting the extending line of B_1A at P_1. Draw arc $\widehat{P_1Q_1}$ with B_1, B_1P_1 as the center and radius respectively, intercepting the extending line of B_1C_0 at Q_1. Draw arc $\widehat{Q_1P_0'}$ with C_0, C_0Q_1 as the center and radius respectively, intercepting the extending line of AB_0 at P_0'. Prove that

(1) P_0' and P_0 are coincident, and arcs $\widehat{P_0Q_0}$ and $\widehat{P_0Q_1}$ are tangent to each other at P_0.

(2) Points P_0, Q_0, Q_1, P_1 are concyclic.

Proof (1) From the properties of an ellipse we know

$$B_1C_0 + C_0B_0 = B_1C_1 + C_1B_0.$$

Also, it is obvious that

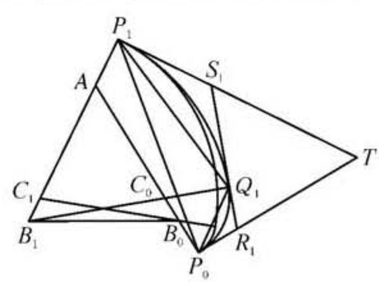
$$B_0P_0 = B_0Q_0,\ C_1B_0 + B_0Q_0 = C_1P_1,$$

$$B_1C_1 + C_1P_1 = B_1C_0 + C_0Q_1,\ C_0Q_1 = C_0B_0 + B_0P_0'.$$

Adding these equations, we get $B_0P_0 = B_0P_0'$.

Therefore P_0' and P_0 are coincident. Furthermore, as P_0, C_0 (the center of $\widehat{Q_1P_0}$) and B_0 (the center of $\widehat{P_0Q_0}$) are lying on the same line, we know that $\widehat{Q_1P_0}$ and $\widehat{P_0Q_0}$ are tangent at P_0.

(2) We have thus $\widehat{Q_1P_0}$ and $\widehat{P_0Q_0}$, $\widehat{P_0Q_0}$ and $\widehat{Q_0P_1}$, $\widehat{Q_0P_1}$ and $\widehat{P_1Q_1}$, $\widehat{P_1Q_1}$ and $\widehat{Q_1P_0'}$ are tangent at points P_0, Q_0, P_1, Q_1 respectively. Now we draw common tangent lines P_0T and P_1T through P_0 and P_1 respectively, and suppose the two line meet at point T. Also, we draw a common



tangent line R_1S_1 through Q_1, and suppose it intercepts P_0T and P_1T at point R_1 and S_1 respectively. Drawing segments P_0Q_1 and P_1Q_1, we get isosceles triangles $P_0Q_1R_1$ and $P_1Q_1S_1$ respectively. Then we have

$$\begin{aligned}\angle P_0Q_1P_1 &= \pi - \angle P_0Q_1R_1 - \angle P_1Q_1S_1 \\ &= \pi - (\angle P_1P_0T - \angle Q_1P_0P_1) \\ &\quad - (\angle P_0P_1T - \angle Q_1P_1P_0).\end{aligned}$$

Since

$$\pi - \angle P_0Q_1P_1 = \angle Q_1P_0P_1 + \angle Q_1P_1P_0,$$

we obtain

$$\angle P_0Q_1P_1 = \pi - \frac{1}{2}(\angle P_1P_0T + \angle P_0P_1T).$$

In the same way, we can prove that

$$\angle P_0Q_0P_1 = \pi - \frac{1}{2}(\angle P_1P_0T + \angle P_0P_1T).$$

It implies that points P_0, Q_0, Q_1, P_1 are concyclic.

2 Suppose an infinite sequence $\{a_n\}$ satisfies $a_0 = x$, $a_1 = y$, $a_{n+1} = \dfrac{a_na_{n-1}+1}{a_n+a_{n-1}}$, $n = 1, 2, \cdots$.

(1) Find all real numbers x and y that satisfy the statement: there exists a positive integer n_0, such that, for $n \geqslant n_0$, a_n is a constant.

(2) Find an explicit expression for a_n.

Solution (1) We have

$$a_n - a_{n+1} = a_n - \frac{a_na_{n-1}+1}{a_n+a_{n-1}} = \frac{a_n^2-1}{a_n+a_{n-1}},\ n = 1, 2, \cdots. \quad ①$$

If there exists a positive integer n such that $a_{n+1} = a_n$, we get

$a_n^2 = 1$ and $a_n + a_{n-1} \neq 0$.

If $n = 1$, we have

$$|y| = 1 \quad \text{and} \quad x \neq -y. \tag{2}$$

If $n > 1$, then

$$a_n - 1 = \frac{a_{n-1}a_{n-2} + 1}{a_{n-1} + a_{n-2}} - 1 = \frac{(a_{n-1} - 1)(a_{n-2} - 1)}{a_{n-1} + a_{n-2}},\ n \geqslant 2, \tag{3}$$

and

$$a_n + 1 = \frac{a_{n-1}a_{n-2} + 1}{a_{n-1} + a_{n-2}} + 1 = \frac{(a_{n-1} + 1)(a_{n-2} + 1)}{a_{n-1} + a_{n-2}},\ n \geqslant 2. \tag{4}$$

Multiplying equations ③ and ④, we get

$$a_n^2 - 1 = \frac{a_{n-1}^2 - 1}{a_{n-1} + a_{n-2}} \cdot \frac{a_{n-2}^2 - 1}{a_{n-1} + a_{n-2}},\ n \geqslant 2. \tag{5}$$

From ⑤ we infer that x and y satisfy either ② or

$$|x| = 1 \quad \text{and} \quad y \neq -x. \tag{6}$$

Conversely, if x and y satisfy either ② or ⑥, then $a_n =$ constant when $n \geqslant 2$ and the constant can only be either 1 or -1.

(2) From ③ and ④, we get

$$\frac{a_n - 1}{a_n + 1} = \frac{a_{n-1} - 1}{a_{n-1} + 1} \cdot \frac{a_{n-2} - 1}{a_{n-2} + 1},\ n \geqslant 2. \tag{7}$$

Let $b_n = \frac{a_n - 1}{a_n + 1}$. Then, for $n \geqslant 2$, equation ⑦ becomes

$$\begin{aligned} b_n &= b_{n-1}b_{n-2} = (b_{n-2}b_{n-3})b_{n-2} = b_{n-2}^2\, b_{n-3} \\ &= (b_{n-3}b_{n-4})^2 b_{n-3} = b_{n-3}^3\, b_{n-4}^2 = \cdots. \end{aligned}$$

Then we get

$$\frac{a_n - 1}{a_n + 1} = \left(\frac{y-1}{y+1}\right)^{F_{n-1}} \cdot \left(\frac{x-1}{x+1}\right)^{F_{n-2}},\ n \geqslant 2, \tag{8}$$

here,

$$F_n = F_{n-1} + F_{n-2},\ n \geqslant 2,\ F_0 = F_1 = 1. \tag{9}$$

From ⑨ we get

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right). \tag{10}$$

The range of n in ⑩ can be extended to negative integers. For example, $F_{-1} = 0$, $F_{-2} = 1$. Since ⑧ holds for any $n \geqslant 0$, we get

$$a_n = \frac{(x+1)^{F_{n-2}}(y+1)^{F_{n-1}} + (x-1)^{F_{n-2}}(y-1)^{F_{n-1}}}{(x+1)^{F_{n-2}}(y+1)^{F_{n-1}} - (x-1)^{F_{n-2}}(y-1)^{F_{n-1}}},\ n \geqslant 0, \tag{11}$$

here F_{n-1}, F_{n-2} are determined by ⑩.

3 Solve the following system of equations.

$$\begin{cases} x - y + z - w = 2, \\ x^2 - y^2 + z^2 - w^2 = 6, \\ x^3 - y^3 + z^3 - w^3 = 20, \\ x^4 - y^4 + z^4 - w^4 = 66. \end{cases}$$

Solution Let $p = x + z$, $q = xz$. The second to fourth equations of the system become

$$\begin{aligned} p^2 &= x^2 + z^2 + 2q, \\ p^3 &= x^3 + z^3 + 3pq, \\ p^4 &= x^4 + z^4 + 4p^2q - 2q^2. \end{aligned}$$

Similarly, let $s = y + w$, $t = yw$. The second to fourth equations of the system become

$$\begin{aligned} s^2 &= y^2 + w^2 + 2t, \\ s^3 &= y^3 + w^3 + 3st, \\ s^4 &= y^4 + w^4 + 4s^2t - 2t^2. \end{aligned}$$

Also, the first equation in the system can now be expressed as

$$p = s + 2. \tag{①}$$

Therefore

$$\begin{aligned} p^2 &= s^2 + 4s + 4, \\ p^3 &= s^3 + 6s^2 + 12s + 8, \\ p^4 &= s^4 + 8s^3 + 24s + 32s + 16. \end{aligned}$$

Substituting the expressions of p^2, p^3, p^4 and s^2, s^3, s^4 obtained previously into the original system, we get

$$\begin{aligned} x^2 + z^2 + 2q &= y^2 + w^2 + 2t + 4s + 4, \\ x^3 + z^3 + 3pq &= y^3 + w^3 + 3st + 6s^2 + 12s + 8, \\ x^4 + z^4 + 4p^2q - 2q^2 &= y^4 + w^4 + 4s^2t - 2t^2 + 8s^3 \\ &\quad + 24s + 32s + 16. \end{aligned}$$

Using the second to the fourth equations in the system to simplify the above, we get

$$q = t + 2s - 1, \tag{②}$$

$$pq = st + 2s^2 + 4s - 4, \tag{③}$$

$$2p^2q - q^2 = 2s^2t - t^2 + 4s^3 + 12s^2 + 16s - 25. \tag{④}$$

Substituting ① and ② into ③, we get

$$t = \frac{s}{2} - 1. \tag{⑤}$$

Substituting ⑤ into ②,

$$q = \frac{5}{2}s - 2. \quad ⑥$$

Substituting ①, ⑤, ⑥ into ④, we get $s = 2$. Therefore $t = 0$, $p = 4$, $q = 3$.

Consequently, x, z and y, w are the roots of equations $X^2 - 4X + 3 = 0$ and $Y^2 - 2Y = 0$ respectively. That means

$$\begin{cases} x = 3, \\ z = 1 \end{cases} \text{or} \begin{cases} x = 1, \\ z = 3 \end{cases}$$

and

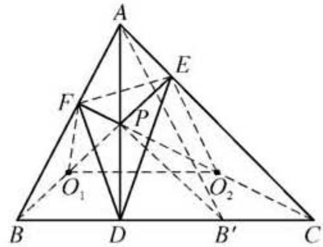
$$\begin{cases} y = 2, \\ w = 0 \end{cases} \text{or} \begin{cases} y = 0, \\ w = 2. \end{cases}$$

Specifically, the system of equations has 4 solutions:

$$x = 3,\ y = 2,\ z = 1,\ w = 0;$$
$$x = 3,\ y = 0,\ z = 1,\ w = 2;$$
$$x = 1,\ y = 2,\ z = 3,\ w = 0;$$
$$x = 1,\ y = 0,\ z = 3,\ w = 2.$$

2007 (Tianjin)

1 As shown in the figure, $\triangle ABC$ is an acute triangle with $AB < AC$. AD is the perpendicular height on BC with point P along AD. Through P draw $PE \perp AC$



with E as foot drop, draw $PF \perp AB$ with F as foot drop. O_1, O_2 are circumcenters of $\triangle BDF$, $\triangle CDE$ respectively. Prove that O_1, O_2, E, F are concyclic if and only if P is the orthocenter of $\triangle ABC$.

Proof Connect BP, CP, O_1O_2, EO_2, EF and FO_1. Since $PD \perp BC$ and $PF \perp AB$, therefore points B, D, P and F are concyclic. BP is the diameter; O_1, being the circumcenter of $\triangle BDF$, is the midpoint of BP. In the same way, C, D, P and E are concyclic, and O_2 is the midpoint of CP. Then $O_1O_2 \parallel BC$, and $\angle PO_2O_1 = \angle PCB$. As

$$AF \cdot AB = AP \cdot AD = AE \cdot AC,$$

we conclude that B, C, E and F are concyclic.

Sufficiency proof. Assume P is the orthocenter of $\triangle ABC$. As $PE \perp AC$ and $PF \perp AB$. We know that points B, O_1, P and E are collinear. Therefore

$$\angle FO_2O_1 = \angle FCB = \angle FEB = \angle FEO_1,$$

and that means O_1, O_2, E and F are concyclic.

Necessity proof. Assume O_1, O_2, E and F are concyclic. Then $\angle O_1O_2E + \angle EFO_1 = 180^\circ$. We have

$$\begin{aligned}\angle O_1O_2E &= \angle O_1O_2P + \angle PO_2E \\ &= \angle PCB + 2\angle ACP \\ &= (\angle ACB - \angle ACP) + 2\angle ACP \\ &= \angle ACB + \angle ACP, \\ \angle EFO_1 &= \angle PFO_1 + \angle PFE \\ &= (90^\circ - \angle ABP) + (90^\circ - \angle ACB).\end{aligned}$$

The last identity holds because B, C, E and F are concyclic. Then

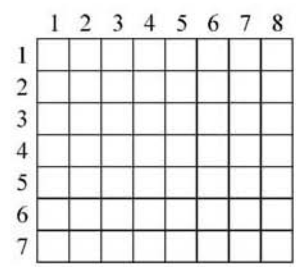
$$\begin{aligned}&\angle O_1O_2E + \angle EFO_1\\ = &\angle ACB + \angle ACP + (90^\circ - \angle ABP) + (90^\circ - \angle ACB)\\ = &180^\circ.\end{aligned}$$

That is, $$\angle ABP = \angle ACP. \quad ①$$

Further, since $AB < AC$ and $AD \perp BC$, then $BD < CD$. There is point B' on CD such that $BD = B'D$. Connecting AB', PB, we have $\angle AB'P = \angle ABP$. By ①, $\angle AB'P = \angle ACP$, and that means A, P, B', C are concyclic. Then $\angle PB'B = \angle CAP = 90^\circ - \angle ACB$, and $\angle PBC + \angle ACB = (90^\circ - \angle ACB) + \angle ACB = 90^\circ$. That means $BP \perp AC$. Therefore P is the orthocenter of $\triangle ABC$.

2 Given a 7×8 checkerboard as seen in the figure, 56 pieces are placed on the board with each square containing exactly one piece. If two pieces share a common side or vertex, they are called "connected". A group of 5 pieces is said to have Property A, if these pieces are connected orderly in a (horizontal, vertical or diagonal) line.

What is the least number of pieces to be removed from the board to ensure that there exists no group of 5 pieces on the board which has Property A? You must prove your answer.



Solution The answer is that at least 11 pieces must be removed. The following is a proof by contradiction.

Assume that removing 10 pieces from the board would satisfy the requirement. We denote the square in row i and column j as (i, j). As shown in Fig. 1, to ensure there are no groups of 5 pieces with Property A, we must remove at least one piece from the first 5 squares of each row, i. e. 7 pieces; then in the last three columns, we must remove at least one piece from the first 5 squares of each column, i. e. 3 pieces. That means pieces in squares $(i, j)(6 \leqslant i \leqslant 7, 6 \leqslant j \leqslant 8)$ are untouched. By symmetry, we conclude that the pieces in the shadowed areas of the four corners of the board as shown in Fig. 1 are untouched while removing 10 pieces satisfying the requirement. Further, it is easy to check that in rows 1, 2, 6, 7 and columns 1, 2, 7, 8, at least one piece should be removed from each row and each column, i. e. 8 pieces. Then at most two of pieces named ①, ②, ③, ④ in Fig. 1 can be removed.

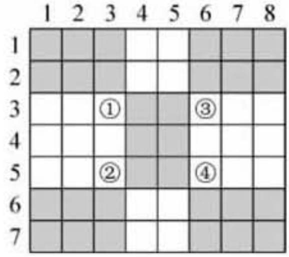


Fig. 1

However, any piece of the four remained will result in a group of 5 pieces with Property A. For example, if piece ① (in square $(3, 3)$) remains, then pieces in $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, $(5, 5)$ are connected orderly in a diagonal line. That means the removal of 10 pieces is impossible to satisfy the requirement.

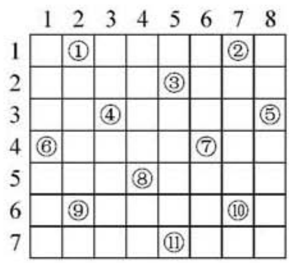


Fig. 2

On the other hand, as shown in Fig. 2, if we remove the 11 pieces in squares numbered ① to ⑪, then there is no group of 5 pieces that remained on the board having Property A. That completes the proof.

3 Given the set $P = \{1, 2, 3, 4, 5\}$, define $f(m, k) = \sum_{i=1}^{5}\left[m\sqrt{\frac{k+1}{i+1}}\right]$ for any $k \in P$ and positive integer m, where $[a]$ denotes the greatest integer less than or equal to a. Prove that for any positive integer n, there is $k \in P$ and positive integer m, such that $f(m, k) = n$.

Proof Define set $A = \{m\sqrt{k+1} \mid m \in \mathbf{N}^*, k \in P\}$, where $\mathbf{N}^*$ denotes the set of all positive integers. It is easy to check that for any $k_1, k_2 \in P$, $k_1 \neq k_2$, $\frac{\sqrt{k_1+1}}{\sqrt{k_2+1}}$ is an irrational number. Therefore, for any $k_1, k_2 \in P$ and positive integers m_1, m_2, $m_1\sqrt{k_1+1} = m_2\sqrt{k_2+1}$ implies $m_1 = m_2$ and $k_1 = k_2$.

Note that A is an infinite set. We arrange the elements in A in ascending order. Then we have an infinite sequence. For any positive integer n, suppose the nth term of the sequence is $m\sqrt{k+1}$. Any term before the nth can be written as $m_i\sqrt{i+1}$, and

$$m_i\sqrt{i+1} \leqslant m\sqrt{k+1}.$$

Or equivalently, $m_i \leqslant m\frac{\sqrt{k+1}}{\sqrt{i+1}}$. It is easy to see that there are $\left[m\frac{\sqrt{k+1}}{\sqrt{i+1}}\right]$ such m_i for $i = 1, 2, 3, 4, 5$. Therefore,

$$n = \sum_{i=1}^{5}\left[m\frac{\sqrt{k+1}}{\sqrt{i+1}}\right] = f(m, k).$$

The proof is complete.

China Mathematical Olympiad

2007 (Wenzhou, Zhejiang)

2007 China Mathematical Olympiad (CMO) and the 22nd Mathematics Winter Camp was held 25 – 30 January 2007 in Wenzhou, Zhejiang Province, and was hosted by the CMO Committee and Wenzhou No. 1 middle school.

The Competition Committee comprises: Leng Gangsong, Zhu Huawei, Chen yonggao, Li Shenghong, Li Weigu, Xiong Bin, Wang Jianwei, Liu Zhipeng, Luo Wei, and Su Chun.

First Day

0800 - 1230 January 27, 2007

1 Given complex numbers a, b, c, let $|a+b|=m$, $|a-b|=n$, and suppose $mn \neq 0$. Prove that

$$\max\{|ac+b|, |a+bc|\} \geqslant \frac{mn}{\sqrt{m^2+n^2}}.$$

Proof Ⅰ We have

$$\begin{aligned}\max\{|ac+b|, |a+bc|\} &\geqslant \frac{|b|\cdot|ac+b|+|a|\cdot|a+bc|}{|b|+|a|}\\ &\geqslant \frac{|b(ac+b)-a(a+bc)|}{|a|+|b|}\\ &= \frac{|b^2-a^2|}{|a|+|b|}\\ &\geqslant \frac{|b+a|\cdot|b-a|}{\sqrt{2(|a^2|+|b^2|)}}.\end{aligned}$$

As

$$m^2+n^2=|a-b|^2+|a+b|^2=2(|a|^2+|b|^2),$$

we get

$$\max\{|ac+b|, |a+bc|\} \geqslant \frac{mn}{\sqrt{m^2+n^2}}.$$

Proof Ⅱ Note that

$$ac+b=\frac{1+c}{2}(a+b)-\frac{1-c}{2}(a-b)$$

and

$$a+bc=\frac{1+c}{2}(a+b)+\frac{1-c}{2}(a-b).$$

Let $\alpha = \frac{1+c}{2}(a+b)$ and $\beta = \frac{1-c}{2}(a-b)$. Then

$$\begin{aligned} |ac+b|^2 + |a+bc|^2 &= |\alpha - \beta|^2 + |\alpha + \beta|^2 \\ &= 2(|\alpha|^2 + |\beta|^2). \end{aligned}$$

Therefore

$$\begin{aligned} (\max\{|ac+b|, |a+bc|\})^2 &\geqslant |\alpha|^2 + |\beta|^2 \\ &= \left|\frac{1+c}{2}\right|^2 m^2 + \left|\frac{1-c}{2}\right|^2 n^2. \end{aligned}$$

Now we only need to prove that

$$\left|\frac{1+c}{2}\right|^2 m^2 + \left|\frac{1-c}{2}\right|^2 n^2 \geqslant \frac{m^2n^2}{m^2+n^2},$$

or equivalently

$$\left|\frac{1+c}{2}\right|^2 m^4 + \left|\frac{1-c}{2}\right|^2 n^4 + \left(\left|\frac{1+c}{2}\right|^2 + \left|\frac{1-c}{2}\right|^2\right) m^2n^2 \geqslant m^2n^2.$$

We have

$$\begin{aligned} &\left|\frac{1+c}{2}\right|^2 m^4 + \left|\frac{1-c}{2}\right|^2 n^4 + \left(\left|\frac{1+c}{2}\right|^2 + \left|\frac{1-c}{2}\right|^2\right) m^2n^2 \\ &\geqslant 2\left|\frac{1+c}{2}\right|\left|\frac{1-c}{2}\right| m^2n^2 + \left(\left|\frac{1+2c+c^2}{4}\right| + \left|\frac{1-2c+c^2}{4}\right|\right) m^2n^2 \\ &= \left(\left|\frac{1-c^2}{2}\right| + \left|\frac{1+2c+c^2}{4}\right| + \left|\frac{1-2c+c^2}{4}\right|\right) m^2n^2 \\ &\geqslant \left|\frac{1-c^2}{2} + \frac{1+2c+c^2}{4} + \frac{1-2c+c^2}{4}\right| m^2n^2 \\ &= m^2n^2. \end{aligned}$$

This completes the proof.

Proof Ⅲ Since

$$\begin{aligned}m^2 &= |a+b|^2 = (a+b)(\overline{a+b}) = (a+b)(\bar{a}+\bar{b}) \\ &= |a^2|+|b^2|+a\bar{b}+\bar{a}b, \\ n^2 &= |a-b|^2 = (a-b)(\overline{a-b}) = (a-b)(\bar{a}-\bar{b}) \\ &= |a^2|+|b^2|-a\bar{b}-\bar{a}b,\end{aligned}$$

We get

$$|a|^2+|b|^2 = \frac{m^2+n^2}{2},$$

$$a\bar{b}+\bar{a}b = \frac{m^2-n^2}{2}.$$

Let $c = x+yi,\ x,\ y \in \mathbf{R}$. Then

$$\begin{aligned}&|ac+b|^2+|a+bc|^2 \\ &= (ac+b)(\overline{ac+b})+(a+bc)(\overline{a+bc}) \\ &= |a|^2\,|c|^2+|b|^2+a\bar{b}c+\bar{a}b\bar{c}+|a|^2 \\ &\quad +|b|^2\,|c|^2+\bar{a}bc+a\,\bar{b}\,\bar{c} \\ &= (|c|^2+1)(|a|^2+|b|^2)+(c+\bar{c})(a\bar{b}+\bar{a}b) \\ &= (x^2+y^2+1)\,\frac{m^2+n^2}{2}+2x\,\frac{m^2-n^2}{2} \\ &\geqslant \frac{m^2+n^2}{2}x^2+(m^2-n^2)x+\frac{m^2+n^2}{2} \\ &= \frac{m^2+n^2}{2}\left(x+\frac{m^2-n^2}{m^2+n^2}\right)^2 \\ &\quad -\frac{m^2+n^2}{2}\left(\frac{m^2-n^2}{m^2+n^2}\right)^2+\frac{m^2+n^2}{2} \\ &\geqslant \frac{m^2+n^2}{2}-\frac{1}{2}\,\frac{(m^2-n^2)^2}{m^2+n^2} \\ &= \frac{2m^2n^2}{m^2+n^2}.\end{aligned}$$

That is

$$(\max\{|ac+b|,\ |a+bc|\})^2 \geqslant \frac{m^2n^2}{m^2+n^2}.$$

Or

$$\max\{|ac+b|, |a+bc|\} \geqslant \frac{mn}{\sqrt{m^2+n^2}}.$$

2 Prove the following statements:

(1) If $2n-1$ is a prime number, then for any group of distinct positive integers $a_1, a_2, \cdots, a_n$ there exist i, $j \in \{1, 2, \cdots, n\}$ such that $\dfrac{a_i+a_j}{(a_i, a_j)} \geqslant 2n-1$.

(2) If $2n-1$ is a composite number, then there exists a group of distinct positive integers $a_1, a_2, \cdots, a_n$ such that $\dfrac{a_i+a_j}{(a_i, a_j)} < 2n-1$ for any i, $j \in \{1, 2, \cdots, n\}$.

Here (x, y) denotes the greatest common divisor of positive integers x and y.

Proof (1) Let $p = 2n-1$ be a prime. Without loss of generality, we assume that $(a_1, a_2, \cdots, a_n) = 1$. If there exists $i (1 \leqslant i \leqslant n)$ such that $p \mid a_i$, then there exists $j (\neq i)$ such that $p \nmid a_j$. Therefore $p \nmid (a_i, a_j)$. Then we have

$$\frac{a_i+a_j}{(a_i, a_j)} \geqslant \frac{a_i}{(a_i, a_j)} \geqslant p = 2n-1.$$

Next, we consider the case when $(a_i, p) = 1$, $i = 1, 2, \cdots, n$. Then $p \nmid (a_i, a_j)$ for any $i \neq j$. By the Pigeonhole Principle, we know that there exist $i \neq j$ such that either $a_i \equiv a_j \pmod p$ or $a_i + a_j \equiv 0 \pmod p$.

Case $a_i \equiv a_j \pmod p$, we have

$$\frac{a_i+a_j}{(a_i, a_j)} \geqslant \frac{a_i-a_j}{(a_i, a_j)} \geqslant p = 2n-1.$$

Case $a_i + a_j \equiv 0 \pmod{p}$, we have

$$\frac{a_i + a_j}{(a_i, a_j)} \geqslant p = 2n - 1.$$

That completes the proof of (1).

(2) We will construct an example to justify the statement. Since $2n - 1$ is a composite number, we can write $2n - 1 = pq$ where p, q are positive integers greater than 1. Let

$$a_1 = 1,\ a_2 = 2,\ \cdots,\ a_p = p,\ a_{p+1} = p + 1,$$
$$a_{p+2} = p + 3,\ \cdots,\ a_n = pq - p.$$

Note that the fist p elements are consecutive integers, while the remainders are $n - p$ consecutive even integers from $p + 1$ to $pq - p$.

When $1 \leqslant i \leqslant j \leqslant p$, it is obvious that

$$\frac{a_i + a_j}{(a_i, a_j)} \leqslant a_i + a_j \leqslant 2p < 2n - 1.$$

When $p + 1 \leqslant i \leqslant j \leqslant n$, we have $2 \mid (a_i, a_j)$ and then

$$\frac{a_i + a_j}{(a_i, a_j)} \leqslant \frac{a_i + a_j}{2} \leqslant pq - p < 2n - 1.$$

When $1 \leqslant i \leqslant p$ and $p + 1 \leqslant j \leqslant n$ we have the following two possibilities:

(i) Either $i \neq p$ or $j \neq n$. Then we have

$$\frac{a_i + a_j}{(a_i, a_j)} \leqslant pq - 1 < 2n - 1.$$

(ii) $i = p$ and $j = n$. Then

$$\frac{a_i + a_j}{(a_i, a_j)} = \frac{pq}{p} = q < 2n - 1.$$

That completes the proof.

3 Let a_1, a_2, $\cdots$, a_{11} be 11 distinct positive integers with their sum less than 2007, and write the numbers 1, 2, $\cdots$, 2007 in order on the blackboard. Now we define a group of 22 ordered operations:

The ith operation is to take any number on the blackboard, and then add a_i to it, if $1 \leqslant i \leqslant 11$ or minus a_{i-11} from it, if $12 \leqslant i \leqslant 22$.

If the final result after such a group of operations is an even permutation of 1, 2, $\cdots$, 2007, then we call it a "good" group; if the result is an odd permutation of 1, 2, $\cdots$, 2007, then we call it a "second good" group.

Our question is: Which is greater? The number of "good" groups or that of "second good" groups? And by how many more?

(**Remark** Suppose x_1, x_2, $\cdots$, x_n is a permutation of 1, 2, $\cdots$, n. We call it an even permutation if $\prod_{i>j}(x_i - x_j) > 0$, and otherwise an odd permutation.)

Solution The answer is: The "good" groups is more than the "second good" groups by $\prod_{i=1}^{11} a_i$.

More generally, we write numbers 1, 2, $\cdots$, n in order on the blackboard, and define a group of l ordered operations: The ith operation is to take any number on the blackboard, and then add b_i ($b_i \in \mathbf{Z}$, $1 \leqslant i \leqslant l$) to it.

If the final result after such a group of operations is an even/odd permutation of 1, 2, $\cdots$, n, then we call it a "good"/"second good" group. And the difference between the number of "good" groups and that of "second good" groups is defined as

$f(b_1, b_2, \cdots, b_l; n)$. Now let us study the property of f.

Firstly, interchanging b_i and b_j for any $1 \leqslant i, j \leqslant l$ will not affect the value of f. As a matter of fact, it only results in the exchange of the ith and jth operations in a group, and will not affect the final result after the group's operations. So the value of f remains the same.

Secondly, we only need to count the number of "good"/"second good" groups with property P — a property attributed to any operation group which keeps the numbers on the blackboard distinctive from one another after each operation. We can prove that the difference between the numbers of "good" and "second good" groups with property P is also equal to f.

In fact, we only need to prove that the numbers of "good" and "second good" groups without property P are the same. Suppose the ith operation of a "good"/"second good" group without property P results in the equal between the pth and qth number on the blackboard ($1 \leqslant p < q \leqslant n$). We change the following $l - i$ operations in this way: operations on the pth number are changed to operations on the qth number, and vice versa. It is easy to verify that the resulted permutation on the blackboard of new operation group would be a (p, q) transposition of the permutation of the original operation group. Then the parities of the two permutations are in opposite signs. And that means the numbers of "good" and "second good" groups without property P are the same.

Now, let $a_1, a_2, \cdots, a_m$ be m distinct positive integers with their sum less than n. We prove by the principle of mathematical induction that

$$f(a_1, a_2, \cdots, a_m, -a_1, -a_2, \cdots, -a_m; n) = \prod_{i=1}^{m} a_i. \quad ①$$

If $m = 1$, consider a "good"/"second good" group with property P. It must be in such a way: The first operation is to take a number from $n - a_1 + 1$, $n - a_2 + 2$, $\cdots$, n on the blackboard, and add a_1 to it; next operation is to add $-a_1$ again to it. So the number of "good" groups is a_1, while that of "second good" groups is 0. Therefore ① holds.

Assume that ① holds for $m - 1$. We now consider case m. According to what discussed above, we may assume that $a_1 < a_2 < \cdots < a_n$, and

$$\begin{aligned} & f(a_1, a_2, \cdots, a_m, -a_1, -a_2, \cdots, -a_m; n) \\ = & f(a_1, -a_2, -a_3, \cdots, -a_m, a_2, a_3, \cdots, a_m, -a_1; n). \end{aligned}$$

For a group with property P, the first operation must be done on the last a_1 numbers on the blackboard; the second operation be done on the first a_2 numbers; the third operation done on the first $a_2 + a_3$ numbers; ... the mth operation done on the first $a_2 + \cdots + a_m < n - a_1$ numbers. And the $m + 1 \sim 2m - 1$ th operations will also be done on the first $n - a_1$ numbers. Otherwise the sum of the first $n - a_1$ numbers will be less than $1 + 2 + \cdots + (n - a_1)$, a contradiction to property P.

Therefore, the $2 \sim 2m - 2$ operations must be done on the first $n - a_1$ numbers, and the result must be an even/odd permutation of $1, 2, \cdots, (n - a_1)$, which corresponds to each one of a_1 even/odd permutations of $1, 2, \cdots, n$ derived from original operation groups. therefore

$$\begin{aligned} & f(a_1, -a_2, -a_3, \cdots, -a_m, a_2, a_3, \cdots, a_m, -a_1; n) \\ = & a_1 f(-a_2, -a_3, \cdots, -a_m, a_2, a_3, \cdots, a_m; n - a_1). \end{aligned}$$

By induction we have

$$\begin{aligned} & f(-a_2, -a_3, \cdots, -a_m, a_2, a_3, \cdots, a_m; n-a_1) \\ = & f(a_2, a_3, \cdots, a_m, -a_2, -a_3, \cdots, -a_m; n-a_1) \\ = & \prod_{j=2}^{m} a_j. \end{aligned}$$

That means ① holds for m.

Now take $n = 2007$ and $m = 11$ in ①. Thus arriving at the value of $\prod_{j=1}^{11} a_j$.

Second Day

0800 - 1200 January 28, 2007

4 Suppose points O and I are the circumcenter and incenter of $\triangle ABC$ respectively, and the inscribed circle of $\triangle ABC$ is tangent to the sides BC, CA, AB at points D, E, F respectively. Lines FD and CA intercept at point P, while lines DE and AB intercept at point Q. And points M, N are the midpoint of segments PE, QF respectively. Prove that $OI \perp MN$.

Proof We first consider $\triangle ABC$ and segment PFD. By *Menelaus* theorem we have

$$\frac{CP}{PA} \cdot \frac{AF}{FB} \cdot \frac{BD}{DC} = 1.$$

Then

$$PA = CP \cdot \frac{AF}{FB} \cdot \frac{BD}{DC} = (PA + b) \frac{p-a}{p-c}.$$

(We define $a = BC$, $b = CA$, $c = AB$, $p = \frac{1}{2}(a+b+c)$; and without loss of generality, assume $a > c$.) Then we get

$$PA = \frac{b(p-a)}{a-c}.$$

Further,

$$PE = PA + AE = \frac{b(p-a)}{a-c} + p - a = \frac{2(p-c)(p-a)}{a-c},$$

$$ME = \frac{1}{2}PE = \frac{(p-c)(p-a)}{a-c},$$

$$MA = ME - AE = \frac{(p-c)(p-a)}{a-c} - (p-a) = \frac{(p-a)^2}{a-c},$$

$$MC = ME + EC = \frac{(p-c)(p-a)}{a-c} + (p-c) = \frac{(p-c)^2}{a-c}.$$

Then we have

$$MA \cdot MC = ME^2.$$

That means ME^2 is equal to the power of M with respect to the circumscribed circle of $\triangle ABC$. Further, since ME is the length of the tangent from M to the inscribed circle of $\triangle ABC$, so ME^2 is also the power of M with respect to the inscribed circle. Hence, M is on the radical axis of the circumscribed and inscribed circles of $\triangle ABC$.

In the same way, N is also on the radical axis. Since the radical axis is perpendicular to OI, then $OI \perp MN$. That completes the proof.

5 Suppose a bounded number sequence $\{a_n\}$ satisfies

$$a_n < \sum_{k=n}^{2n+2006} \frac{a_k}{k+1} + \frac{1}{2n+2007}, \quad n = 1, 2, 3, \cdots.$$

Prove that $a_n < \frac{1}{n}$, $n = 1, 2, 3, \cdots$.

Proof Let $b_n = a_n - \frac{1}{n}$. It is routine to check that

$$b_n < \sum_{k=n}^{2n+2006} \frac{b_k}{k+1}, \ n = 1, 2, 3, \cdots. \tag{①}$$

We will prove that $b_n < 0$. As $\{a_n\}$ is bounded, then there exists M such that $b_n < M$. When $n > 100\,000$, we have

$$\begin{aligned} b_n &< \sum_{k=n}^{2n+2006} \frac{b_k}{k+1} \\ &< M \sum_{k=n}^{2n+2006} \frac{1}{k+1} \\ &= M \sum_{k=n}^{\left[\frac{3n}{2}\right]} \frac{1}{k+1} + M \sum_{k=\left[\frac{3n}{2}\right]+1}^{2n+2006} \frac{1}{k+1} \\ &< M \cdot \frac{1}{2} + M \cdot \frac{\frac{n}{2} + 2006}{\frac{3n}{2} + 1} \\ &< \frac{6}{7} M, \end{aligned}$$

where $[x]$ is the greatest integer less than or equal to x.

We can substitute $\frac{6}{7}M$ for M, and go on with the previous steps. Then for any $m \in \mathbf{N}$ we have

$$b_n < \left(\frac{6}{7}\right)^m M,$$

which implies that $b_n \leqslant 0$ for $n \geqslant 100\,000$. Substitute it into ① we get $b_n < 0$ for $n \geqslant 100\,000$.

We observe in ① that, if for any $n \geqslant N + 1$, $b_n < 0$ then

$b_N < 0$. That means

$$b_n < 0 \text{ for } n = 1, 2, 3, \cdots.$$

This implies that $a_n < \frac{1}{n}$, $n = 1, 2, 3, \cdots$.

The proof is complete.

6 Find the smallest positive integer $n \geqslant 9$ satisfying that for any group of integers $a_1, a_2, \cdots, a_n$, there always exist $a_{i_1}, a_{i_2}, \cdots, a_{i_9}$ ($1 \leqslant i_1 < i_2 < \cdots < i_9 \leqslant n$) and $b_i \in \{4, 7\}$ ($i = 1, 2, \cdots, 9$) such that $b_1 a_{i_1} + b_2 a_{i_2} + \cdots + b_9 a_{i_9}$ is a multiple of 9.

Solution Let $a_1 = a_2 = 1$, $a_3 = a_4 = 3$, $a_5 = \cdots = a_{12} = 0$. It is easy to check that any 9 integers of them will not meet the requirement. So $n \geqslant 13$. We will prove that $n = 13$.

We only need to prove the following statement:

Given a group of m integers $a_1, a_2, \cdots, a_m$, if there are not three $a_{i_1}, a_{i_2}, a_{i_3}$ in them and $b_1, b_2, b_3 \in \{4, 7\}$ such that $b_1 a_{i_1} + b_2 a_{i_2} + b_3 a_{i_3} \equiv 0 \pmod 9$, then either $m \leqslant 6$ or $7 \leqslant m \leqslant 8$ and there are $a_{i_1}, a_{i_2}, \cdots, a_{i_6}$ in $a_1, a_2, \cdots, a_m$ and $b_1, b_2, \cdots, b_6 \in \{4, 7\}$ such that $9 \mid b_1 a_{i_1} + b_2 a_{i_2} + \cdots + b_6 a_{i_6}$.

We define

$$A_1 = \{i \mid 1 \leqslant i \leqslant m, 9 \mid a_i\},$$
$$A_2 = \{i \mid 1 \leqslant i \leqslant m, a_i \equiv 3 \pmod 9\},$$
$$A_3 = \{i \mid 1 \leqslant i \leqslant m, a_i \equiv 6 \pmod 9\},$$
$$A_4 = \{i \mid 1 \leqslant i \leqslant m, a_i \equiv 1 \pmod 3\},$$
$$A_5 = \{i \mid 1 \leqslant i \leqslant m, a_i \equiv 2 \pmod 3\}.$$

Then $|A_1| + |A_2| + |A_3| + |A_4| + |A_5| = m$ and

(1) if $i \in A_2$, $j \in A_3$ then $9 \mid 4a_i + 4a_j$;

(2) if $i \in A_4$, $j \in A_5$ then one of $4a_i + 4a_j$, $4a_i + 7a_j$ and $7a_i + 4a_j$ is a multiple of 9 as all of them are divisible by 3 and they are distinct according to mod 9;

(3) if either $i, j, k \in A_2$ or $i, j, k \in A_3$ then $9 \mid 4a_i + 4a_j + 4a_k$;

(4) if either $i, j, k \in A_4$ or $i, j, k \in A_5$ then one of $4a_i + 4a_j + 4a_k$, $4a_i + 4a_j + 7a_k$ and $4a_i + 7a_j + 7a_k$ is a multiple of 9 as all of them are divisible by 3 and they are distinctive according to mod 9.

By the assumption, we have $|A_i| \leqslant 2$ $(1 \leqslant i \leqslant 5)$.

If $|A_1| \geqslant 1$, then $|A_2| + |A_3| \leqslant 2$, $|A_4| + |A_5| \leqslant 2$. Hence

$$m = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| \leqslant 6.$$

Now assume $|A_1| = 0$ and $m \geqslant 7$. Then

$$7 \leqslant m = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| \leqslant 8.$$

Further,

$$\min\{|A_2|, |A_3|\} + \min\{|A_4|, |A_5|\} \geqslant 3.$$

From (1) and (2) we know there exist $i_1, i_2, \cdots, i_6 \in A_2 \cup A_3 \cup A_4 \cup A_5$ $(i_1 < i_2 < \cdots < i_6)$ and $b_1, b_2, \cdots, b_6 \in \{4, 7\}$ such that $9 \mid b_1 a_{i_1} + b_2 a_{i_2} + \cdots + b_6 a_{i_6}$.

The proof of statement is complete.

Now, when $n \geqslant 13$ it is easy to verify with the statement, for any group of integers $a_1, a_2, \cdots, a_n$, there always exist a_{i_1}, a_{i_2}, $\cdots$, a_{i_9} $(1 \leqslant i_1 < i_2 < \cdots < i_9 \leqslant n)$ and $b_i \in \{4, 7\}$ $(i = 1, 2, \cdots, 9)$ such that $b_1 a_{i_1} + b_2 a_{i_2} + \cdots + b_9 a_{i_9}$ is a multiple of 9. That completes the proof.

2008 (Harbin, Heilongjiang)

First Day

0800 - 1230 January 19, 2008

1 Let $\triangle ABC$ be a non-isosceles acute triangle, and point O is the circumcenter. Let A' be a point on the line AO such that $\angle BA'A = \angle CA'A$. Construct $A'A_1 \perp AC$, $A'A_2 \perp AB$ with A_1 on AC, A_2 on AB respectively. AH_A is perpendicular to BC at H_A. Write R_A as the circumradius of $\triangle H_AA_1A_2$. Similarly we have R_B, R_C. Prove that

$$\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}=\frac{2}{R},$$

where R is the circumradius of $\triangle ABC$.

Proof Firstly we claim that A', B, O, C are concyclic points. Otherwise, extend AO to intersect the circumcircle of $\triangle BOC$ at point P which is different from A'. We get

$$\angle BPA = \angle BCO = \angle CBO = \angle CPA.$$

Then $\triangle PA'C \cong \triangle PA'B$, and $A'B = A'C$. So $AB = AC$ and that is a contradiction since $\triangle ABC$ is not isosceles. So

$$\angle BCA' = \angle BOA' = 180^\circ - 2\angle C,$$

and $\angle A'CA_1 = \angle C$.

Further, we have

$$\frac{H_AA}{AC} = \sin\angle C = \cos\angle A_2AA' = \frac{AA_2}{AA'},$$

and

$$\angle A'AC = \angle A_2AH_A = \frac{\pi}{2} - \angle B.$$

So $\triangle A_2AH_A \backsim \triangle A'AC$. In the same way, $\triangle A_1H_AA \backsim \triangle A'BA$. Then $\angle A_2H_AA = \angle ACA'$ and $\angle A_1H_AA = \angle ABA'$. Consequently,

$$\begin{aligned}\angle A_1H_AA_2 &= 2\pi - \angle A_2H_AA - \angle A_1H_A A \\ &= 2\pi - \angle ACA' - \angle ABA' \\ &= \angle A + 2\left(\frac{\pi}{2} - \angle A\right) \\ &= \pi - \angle A.\end{aligned}$$

We get

$$\frac{R}{R_A} = \frac{R}{\dfrac{A_1A_2}{2\sin \angle A_1H_AA_2}} = \frac{2R}{\dfrac{A_1A_2}{\sin \angle A}} = \frac{2R}{AA'}; \qquad ①$$

The last equality holds since A, A_2, A', A_1 lie on the same circle with AA' as the diameter.

Now, draw $AA'' \perp A'C$ with point A'' on line $A'C$. Since $\angle ACA'' = \angle A'CA_1 = \angle C$, we have $AA'' = AH_A$. Then

$$\begin{aligned}AA' &= \frac{AA''}{\sin \angle AA'C} = \frac{AH_A}{\sin(90^\circ - \angle A)} \\ &= \frac{AH_A}{\cos \angle A} = \frac{2S_{\triangle ABC}}{BC\cos \angle A}.\end{aligned} \qquad ②$$

From ①, ② we get

$$\begin{aligned}\frac{1}{R_A} &= \frac{BC\cos \angle A}{S_{\triangle ABC}} = \frac{\cos \angle A}{R\sin \angle B \sin \angle C} \\ &= \frac{1}{R}(1 - \cot \angle B \cot \angle C).\end{aligned}$$

In the same way

$$\frac{1}{R_B} = \frac{1}{R}(1 - \cot \angle C \cot \angle A)$$

and

$$\frac{1}{R_C} = \frac{1}{R}(1 - \cot \angle A \cot \angle B).$$

Notice that

$$\cot\angle A \cot\angle B + \cot\angle B \cot\angle C + \cot\angle C \cot\angle A = 1.$$

We then have

$$\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} = \frac{2}{R}.$$

This completes the proof.

2 Given an integer $n \geqslant 3$, prove that the set $X = \{1, 2, 3, \cdots, n^2 - n\}$ can be divided into two non-intersecting subsets such that neither of them contains n elements a_1, a_2, $\cdots$, a_n with $a_1 < a_2 < \cdots < a_n$ and $a_k \leqslant \frac{a_{k-1} + a_{k+1}}{2}$ for all $k = 2, \cdots, n-1$.

Proof Define

$$S_k = \{k^2 - k + 1, k^2 - k + 2, \cdots, k^2\},$$

$$T_k = \{k^2 + 1, k^2 + 2, \cdots, k^2 + k\}.$$

Let $S = \bigcup_{k=1}^{n-1} S_k$, $T = \bigcup_{k=1}^{n-1} T_k$. We will prove that S, T are the required subsets of X.

Firstly it is easy to verify that $S \cap T = \varnothing$ and $S \cup T = X$.

Next we suppose for contradiction that S contains elements a_1, a_2, $\cdots$, a_n with $a_1 < a_2 < \cdots < a_n$ and $a_k \leqslant \frac{a_{k-1} + a_{k+1}}{2}$ for $k = 2, \cdots, n-1$. Then we have

$$a_k - a_{k-1} \leqslant a_{k+1} - a_k,\ k = 2, \cdots, n-1. \qquad ①$$

Assume that $a_1 \in S_i$, we have $i < n-1$, since $|S_{n-1}| < n$. There exists at least $n - |S_i| = n - i$ elements in $\{a_1, a_2, \cdots, a_n\} \cap (S_{i+1} \cup \cdots \cup S_{n-1})$. Applying the pigeonhole principle, we get that there is an $S_j\ (i < j < n)$ which contains at least two elements in $a_1, a_2, \cdots, a_n$. That means there exists a_k such that $a_k, a_{k+1} \in S_j$ and $a_{k-1} \in S_1 \cup \cdots \cup S_{j-1}$.

Then we have

$$a_{k+1} - a_k \leqslant |S_j| - 1 = j - 1,\ a_k - a_{k-1} \geqslant |T_{j-1}| + 1 = j.$$

That means $a_{k+1} - a_k < a_k - a_{k-1}$, contradicting ①.

In the same way, we can prove that T does not contain $a_1, a_2, \cdots, a_n$ with the required properties either. This completes the proof.

3 Given an integer $n > 0$ and real numbers $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$, $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$, satisfying $\sum_{i=1}^{n} i x_i = \sum_{i=1}^{n} i y_i$. Prove that for any real number α, $\sum_{i=1}^{n} x_i [i\alpha] \geqslant \sum_{i=1}^{n} y_i [i\alpha]$, where $[\beta]$ is defined as the greatest integer less than or equal to β.

Proof I We need the following lemma.

Lemma *For any real number α and positive number n, we have*

$$\sum_{i=1}^{n-1} [i\alpha] \leqslant \frac{n-1}{2}[n\alpha]. \qquad ①$$

The lemma is obtained by summing inequalities

$$[i\alpha] + [(n-i)\alpha] \leqslant [n\alpha]$$

for $i = 1, 2, \cdots, n-1$.

Return to the original problem. We will prove it by induction. For $n = 1$, it is obviously true.

Assume that for $n = k$, it is also true. Now consider $n = k + 1$. Let $a_i = x_i + \frac{2}{k}x_{k+1}$, $b_i = y_i + \frac{2}{k}y_{k+1}$ for $i = 1, 2, \cdots, k$. Then we have $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$, $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k$ and $\sum_{i=1}^{k} ia_i = \sum_{i=1}^{k} ib_i$. By induction we get $\sum_{i=1}^{k} a_i[i\alpha] \geqslant \sum_{i=1}^{k} b_i[i\alpha]$.

In addition, $x_{k+1} \geqslant y_{k+1}$. Otherwise, if $x_{k+1} < y_{k+1}$, we have

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{k+1} < y_{k+1} \leqslant \cdots \leqslant y_2 \leqslant y_1.$$

This contradicts $\sum_{i=1}^{k+1} ix_i = \sum_{i=1}^{k+1} iy_i$. So we have

$$\begin{aligned}\sum_{i=1}^{k+1} x_i[i\alpha] - \sum_{i=1}^{k} a_i[i\alpha] &= x_{k+1}\left\{[(k+1)\alpha] - \frac{2}{k}\sum_{i=1}^{k}[i\alpha]\right\} \\ &\geqslant y_{k+1}\left\{[(k+1)\alpha] - \frac{2}{k}\sum_{i=1}^{k}[i\alpha]\right\} \\ &= \sum_{i=1}^{k+1} y_i[i\alpha] - \sum_{i=1}^{k} b_i[i\alpha].\end{aligned}$$

That means $\sum_{i=1}^{k+1} x_i[i\alpha] \geqslant \sum_{i=1}^{k+1} y_i[i\alpha]$.

By induction, we complete the proof for any integer $n > 0$.

Proof Ⅱ Define $z_i = x_i - y_i$ for $i = 1, 2, \cdots, n$, we have $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n$ and $\sum_{i=1}^{n} iz_i = 0$. We only need to prove that

$$\sum_{i=1}^{n} z_i[i\alpha] \geqslant 0. \qquad ②$$

Let $\Delta_1 = z_1$, $\Delta_2 = z_2 - z_1$, $\cdots$, $\Delta_n = z_n - z_{n-1}$. Then $z_i = \sum_{j=1}^{i} \Delta_j (1 \leqslant i \leqslant n)$, and

$$0=\sum_{i=1}^{n} i z_i=\sum_{i=1}^{n} i \sum_{j=1}^{i} \Delta_j=\sum_{j=1}^{n} \Delta_j \sum_{i=j}^{n} i.$$

So we have

$$\Delta_1=-\sum_{j=2}^{n} \Delta_j \sum_{i=j}^{n} i \Big/ \sum_{i=1}^{n} i. \qquad ③$$

Then

$$\begin{aligned}\sum_{i=1}^{n} z_i[i\alpha] &= \sum_{i=1}^{n}[i\alpha] \sum_{j=1}^{i} \Delta_j=\sum_{j=1}^{n} \Delta_j \sum_{i=j}^{n}[i\alpha] \\ &= \sum_{j=2}^{n} \Delta_j \sum_{i=j}^{n}[i\alpha]-\sum_{j=2}^{n} \Delta_j\Big(\sum_{i=j}^{n} i \Big/ \sum_{i=1}^{n} i\Big) \sum_{i=1}^{n}[i\alpha] \\ &= \sum_{j=2}^{n} \Delta_j \sum_{i=j}^{n} i \cdot\Big(\sum_{i=j}^{n}[i\alpha] \Big/ \sum_{i=j}^{n} i-\sum_{i=1}^{n}[i\alpha] \Big/ \sum_{i=1}^{n} i\Big).\end{aligned}$$

Then, in order to prove ② we only need to prove that for any $2 \leqslant j \leqslant n$, the following inequality holds

$$\sum_{i=j}^{n}[i\alpha] \Big/ \sum_{i=j}^{n} i \geqslant \sum_{i=1}^{n}[i\alpha] \Big/ \sum_{i=1}^{n} i. \qquad ④$$

But

$$\begin{aligned}④ &\Leftrightarrow \sum_{i=j}^{n}[i\alpha] \Big/ \sum_{i=j}^{n} i \geqslant \sum_{i=1}^{j-1}[i\alpha] \Big/ \sum_{i=1}^{j-1} i \\ &\Leftrightarrow \sum_{i=1}^{n}[i\alpha] \Big/ \sum_{i=1}^{n} i \geqslant \sum_{i=1}^{j-1}[i\alpha] \Big/ \sum_{i=1}^{j-1} i.\end{aligned}$$

Then we only need to prove, for any $k \geqslant 1$,

$$\sum_{i=1}^{k+1}[i\alpha] \Big/ \sum_{i=1}^{k+1} i \geqslant \sum_{i=1}^{k}[i\alpha] \Big/ \sum_{i=1}^{k} i,$$

that is equivalent to prove

$$\begin{aligned}&[(k+1)\alpha] \cdot k/2 \geqslant \sum_{i=1}^{k}[i\alpha] \\ \Leftrightarrow &\sum_{i=1}^{k}\big([(k+1)\alpha]-[i\alpha]-[(k+1-i)\alpha]\big) \geqslant 0.\end{aligned}$$

Note that, $[x+y] \geqslant [x]+[y]$ holds for any real numbers x, y, hence ④ holds. The proof is complete.

Second Day

0800 - 1230 January 20, 2008

4 Let $n>1$ be a given integer and A be an infinite set of positive integers satisfying: for any prime $p \nmid n$, there exist infinitely many elements of A not divisible by p. Prove that for any integer $m>1$, $(m, n)=1$, there exists a finite subset of A whose sum of elements, say S, satisfies $S \equiv 1 \pmod{m}$ and $S \equiv 0 \pmod{n}$.

Proof Ⅰ Suppose a prime p satisfies $p^{\alpha} \mid m$. Then from the given conditions, there exists an infinite subset A_1 of A such that p is coprime to every element in A_1.

By the pigeonhole principle, there is an infinite subset A_2 of A_1, such that $x \equiv a \pmod{mn}$, for each element $x \in A_2$, where a is a positive integer and $p \nmid a$.

Since $(m, n)=1$, we have $\left(p^{\alpha}, \dfrac{mn}{p^{\alpha}}\right)=1$. By the Chinese Remainder Theorem, We know that

$$\begin{cases} x \equiv a^{-1} \pmod{p^{\alpha}}, \\ x \equiv 0 \left(\bmod \dfrac{mn}{p^{\alpha}}\right) \end{cases} \qquad ①$$

have infinitely many solutions. Among them we take one as x. Next define B_p as the set of the first x elements in A_2, and S_p as the sum of all elements in B_p. Then we have $S_p \equiv ax \pmod{mn}$. By ① we have

$$S_p \equiv ax \equiv 1 \pmod{p^{\alpha}},\ S_p \equiv 0 \left(\bmod \frac{mn}{p^{\alpha}}\right).$$

Suppose that $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, and for every $p_i (1 \leqslant i \leqslant k-1)$ select a finite subset B_i of A, where $B_i \subset A \backslash B_1 \cup \cdots \cup B_{i-1}$, such that S_{p_i}, the sum of all the elements in B_i, satisfies

$$S_{p_i} \equiv 1 \pmod{p_i^{\alpha_i}}, \ S_{p_i} \equiv 0 \left(\bmod \frac{mn}{p_i^{\alpha_i}}\right). \quad ②$$

Let $B = \bigcup_{i=1}^{k} B_i$, whose sum of elements then satisfies $S = \sum_{i=1}^{k} S_i$. According to ②, we have $S \equiv 1 \pmod{p_i^{\alpha_i}}$ $(1 \leqslant i \leqslant k)$, and $S \equiv 0 \pmod{n}$. So B is the required subset, and the proof is complete.

Proof Ⅱ Divide every element in A by mn, and let the remainders which occur infinitely be (in order) $\alpha_1, \alpha_2, \cdots, \alpha_k$. We claim that

$$(\alpha_1, \alpha_2, \cdots, \alpha_k, m) = 1. \quad ③$$

Otherwise, assume that $p \mid (\alpha_1, \alpha_2, \cdots, \alpha_k, m)$. Then $p \nmid n$, since $(m, n) = 1$. By the given conditions we know that there exist infinitely many elements of A not divisible by p. But by the definition of $\alpha_1, \alpha_2, \cdots, \alpha_k$, the number of such elements are finite. Then by contradiction, ③ holds. Consequently, there are $x_1, x_2, \cdots, x_k, y$, satisfying $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k - ym = 1$. Choose a suitable positive integer r such that $rn \equiv 1 \pmod{m}$. Then

$$\alpha_1 (rnx_1) + \alpha_2 (rnx_2) + \cdots + \alpha_k (rnx_k) = rn + rmny.$$

We select in order rnx_i elements from A such that the remainders by mn are $\alpha_i (i = 1, 2, \cdots, k)$. The set of all these elements is acquired.

5 Find the least positive integer n with the following

property: Paint each vertex of a regular n-gon arbitrarily with one of three colors, say red, yellow and blue, there must exist four vertices of the same color that constitute the vertices of some isogonal trapezoid.

Solution We claim that the least positive integer n is 17.

Firstly we prove that $n = 17$ has the required property. By contradiction, assume that we have a painting pattern with three colors for the regular 17-gon such that any group of 4 vertices of the same color cannot constitute an isogonal trapezoid.

As $\left[\frac{17-1}{3}\right]+1=6$, there exists a group of 6 vertices of the same color, say yellow. Connecting these vertices one another with lines, we get $\binom{6}{2}= 15$ segments. Since the lengths of the segments have at most $\left[\frac{17}{2}\right]=8$ variations, one of the following two cases must exist:

(a) There is a group of three segments with the same length. Since $3 \nmid 7$, not every pair of the segments in the group has a common vertex. So there are two segments in the group which have no common vertex. The four vertices of the two segments constitutes an isogonal trapezoid, and this is a contradiction.

(b) There are 7 pairs of segments with the same length. Then each pair must have a common vertex for its segments. Otherwise, the 4 vertices of the segments in a pair with no common vertex will constitute an isogonal trapezoid. On the other hand, by the pigeonhole principle, we know that there are two pairs which share the same vertex as their segments' common vertex. Then another four vertices of the segments in these two pairs constitute an isogonal trapezoid. This leads to a contradiction again. So $n = 17$ has the required property.

Next we will construct painting patterns for $n \leqslant 16$, which do not have the required property. Define A_1, A_2, $\cdots$, A_n as the vertices of a regular n-gon (order in clockwise), and M_1, M_2, M_3 as the sets of vertices with the same color — red, yellow and blue respectively.

When $n = 16$, let

$$M_1 = \{A_5, A_8, A_{13}, A_{14}, A_{16}\},$$
$$M_2 = \{A_3, A_6, A_7, A_{11}, A_{15}\},$$
$$M_3 = \{A_1, A_2, A_4, A_9, A_{10}, A_{12}\}.$$

In M_1, it is easy to check that the distances from A_{14} to the other 4 vertices are different from each other, and the latter 4 vertices constitute a rectangle, not an isogonal trapezoid. Similarly, no 4 vertices in M_2 constitute an isogonal trapezoid either. As to M_3, the 6 vertices in it are just vertices of three diameters. So any group of four vertices that constitutes either a rectangle or a 4-gon with its sides of different lengths.

When $n = 15$, let

$$M_1 = \{A_1, A_2, A_3, A_5, A_8\},$$
$$M_2 = \{A_6, A_9, A_{13}, A_{14}, A_{15}\},$$
$$M_3 = \{A_4, A_7, A_{10}, A_{11}, A_{12}\}.$$

It is easy to check that no four vertices in each M_i $(i = 1, 2, 3)$ that constitute an isogonal trapezoid.

When $n = 14$, let

$$M_1 = \{A_1, A_3, A_8, A_{10}, A_{14}\},$$
$$M_2 = \{A_4, A_5, A_7, A_{11}, A_{12}\},$$
$$M_3 = \{A_2, A_6, A_9, A_{13}\}.$$

This can be verified easily.

When $n = 13$, let

$$\begin{aligned} M_1 &= \{A_5, A_6, A_7, A_{10}\}, \\ M_2 &= \{A_1, A_8, A_{11}, A_{12}\}, \\ M_3 &= \{A_2, A_3, A_4, A_9, A_{13}\}. \end{aligned}$$

This can be easily verified too. As in this case, we drop A_{13} from M_3, then we arrive at the case for $n = 12$; further drop A_{12}, we have the case $n = 11$; and further drop A_{11}, we get the case $n = 10$.

When $n \leqslant 9$, we can construct a painting pattern such that $|M_i| < 4$ $(i = 1, 2, 3)$, to ensure that no four vertices of the same color constitute an isogonal trapezoid.

By now, we have checked all the cases for $n \leqslant 16$. This completes the proof that 17 is the least value for n to have the required property.

6 Find all triples (p, q, n) such that

$$q^{n+2} \equiv 3^{n+2} \pmod{p^n}, \quad p^{n+2} \equiv 3^{n+2} \pmod{q^n}$$

where p, q are positive odd primes and $n > 1$ is an integer.

Solution It is easy to check that $(3, 3, n)$ $(n = 2, 3, \cdots)$ satisfy both equations. Now let (p, q, n) be another triple satisfying the condition. Then we must have $p \neq q$, $p \neq 3$, $q \neq 3$. We may assume that $q > p \geqslant 5$.

If $n = 2$, then $q^2 \mid p^4 - 3^4$, or $q^2 \mid (p^2 - 3^2)(p^2 + 3^2)$. Then either $q^2 \mid p^2 - 3^2$ or $q^2 \mid p^2 + 3^2$, since q cannot divide both $p^2 - 3^2$ and $p^2 + 3^2$. On the other hand, $0 < p^2 - 3^2 < q^2$, $\frac{1}{2}(p^2 + 3^2) < p^2 < q^2$. This leads to a contradiction.

So $n \geqslant 3$. From $p^n \mid q^{n+2} - 3^{n+2}$, $q^n \mid p^{n+2} - 3^{n+2}$, we get

$$p^n \mid p^{n+2}+q^{n+2}-3^{n+2},\ q^n \mid p^{n+2}+q^{n+2}-3^{n+2}.$$

Since $p<q$, and p, q primes, we have

$$p^nq^n \mid p^{n+2}+q^{n+2}-3^{n+2}. \qquad ①$$

Then $p^nq^n \leqslant p^{n+2}+q^{n+2}-3^{n+2}<2q^{n+2}$. That means $p^n<2q^2$.

As $q^n \mid p^{n+2}-3^{n+2}$ and $p>3$, We have $q^n \leqslant p^{n+2}-3^{n+2}<p^{n+2}$, and consequently $q<p^{1+\frac{2}{n}}$. Since $p^n<2q^2$, we have $p^n<2p^{2+\frac{4}{n}}<p^{3+\frac{4}{n}}$. So $n<3+\dfrac{4}{n}$, and we get $n=3$. Then $p^3 \mid q^5-3^5$, $q^3 \mid p^5-3^5$.

From $5^5-3^5=2\times 11\times 131$, we know $p>5$; from $p^3 \mid q^5-3^5$ we know $p \mid q^5-3^5$. By the Fermat's little theorem, we get $p \mid q^{p-1}-3^{p-1}$. Then $p \mid q^{(5,\,p-1)}-3^{(5,\,p-1)}$.

If $(5, p-1)=1$, then $p \mid q-3$. From

$$\begin{aligned}\frac{q^5-3^5}{q-3}&=q^4+q^3\cdot 3+q^2\cdot 3^2+q\cdot 3^3+3^4\\&\equiv 5\times 3^4 \pmod p\end{aligned}$$

and $p\geqslant 5$, we get $p\nmid\dfrac{q^5-3^5}{q-3}$. So $p^3 \mid q-3$. From $q^3 \mid p^5-3^5$, we get $q^3\leqslant p^5-3^5<p^5=(p^3)^{\frac{5}{3}}<q^{\frac{5}{3}}$. This is a contradiction.

So we have $(5, p-1)\neq 1$, and that means $5 \mid p-1$. In a similar way, we have $5 \mid q-1$. As $(q, p-3)=1$ (since $q>p\geqslant 7$) and $q^3 \mid p^5-3^5$, we know that $q^3 \,\Big|\, \dfrac{p^5-3^5}{p-3}$. Then

$$q^3\leqslant\frac{p^5-3^5}{p-3}=p^4+p^3\cdot 3+p^2\cdot 3^2+p\cdot 3^3+3^4.$$

From $5 \mid p-1$ and $5 \mid q-1$, we get $p\geqslant 11$ and $q\geqslant 31$. So

$$q^3\leqslant p^4\left(1+\frac{3}{p}+\left(\frac{3}{p}\right)^2+\left(\frac{3}{p}\right)^3+\left(\frac{3}{p}\right)^4\right)$$

$$< p^4 \cdot \frac{1}{1-\frac{3}{p}} \leqslant \frac{11}{8} p^4.$$

Then we have $p > \left(\frac{8}{11}\right)^{\frac{1}{4}} q^{\frac{3}{4}}$. Consequently,

$$\frac{p^5+q^5-3^5}{p^3 q^3} < \frac{p^2}{q^3} + \frac{q^2}{p^3} < \frac{1}{q} + \left(\frac{11}{8}\right)^{\frac{3}{4}} \frac{1}{31^{\frac{1}{4}}} < 1.$$

But this contradicts ① which says $p^3 q^3 \mid p^5 + q^5 - 3^5$.

So we reach the conclusion that $(3, 3, n)$ $(n = 2, 3, \cdots)$ are all the triples that satisfy the conditions.

China National Team Selection Test

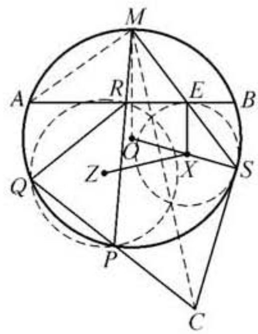
2007 (Shenzhen, Guangdong)

First Day

0800 - 1230 March 31, 2007

1. Let AB be a chord of circle O, M the midpoint of arc AB, and C a point outside of the circle O. From C draw two tangents to the circle at points S, T. $MS \cap AB = E$, $MT \cap AB = F$. From E, F draw a line perpendicular to AB, and intersecting OS, OT at X, Y respectively. Now draw a line from C which intersects the circle O at P and

Q. Let Z be the circumcenter of $\triangle PQR$. Prove that X, Y, Z are collinear.



Proof Refer to the figure, join points O and M. Then OM is the perpendicular bisector of AB. So $\triangle XES \backsim \triangle OMS$, and thus $SX = XE$.

Now draw a circle with center X whose radius is XE. Then the circle X is tangent to chord AB and line CS. Draw the circumcircle of $\triangle PQR$, line MA and line MC.

It is easy to see ($\triangle AMR \backsim \triangle PMA$ etc)

$$MR \cdot MP = MA^2 = ME \cdot MS. \qquad ①$$

By the Power of a Point theorem,

$$CQ \cdot CP = CS^2. \qquad ②$$

So M, C are on the radical axis of circle Z and circle X. Thus

$$ZX \perp MC.$$

Similarly, we have $ZY \perp MC$.

So X, Y, Z are collinear.

2 The rational number x is called "good" if $x = \dfrac{p}{q} > 1$, where p, q are coprime positive integers, and there are α and N such that for every integer $n \geqslant N$,

$$|\{x^n\} - \alpha| \leqslant \frac{1}{2(p+q)},$$

where $\{a\} = a - [a]$, and $[a]$ is the greatest integer less than or equal to a.

Find all "good" rational numbers.

Solution It is obvious that any integer greater than 1 is "good", we will prove that every "good" rational number x (>1) is an integer.

Let $x=\frac{p}{q}>1$ be a "good" number. Denote $m_n=[x^{n+1}]-[x^n]$. Then if $n \geqslant N$, we have

$$\begin{aligned}
&\mid (x-1)x^n-m_n \mid \\
&=\mid \{x^{n+1}\}-\{x^n\} \mid \\
&\leqslant \mid \{x^{n+1}\}-\alpha \mid + \mid \{x^n\}-\alpha \mid \\
&\leqslant \frac{1}{p+q}.
\end{aligned}$$

In view of $(x-1)x^n-m_n=\frac{y}{q^{n+1}}$, where $y \in \mathbf{Z}$, and $\gcd(y, p^{n+1})=1$, So

$$\mid (x-1)x^n-m_n \mid < \frac{1}{p+q}.$$

Thus,

$$\begin{aligned}
&\mid qm_{n+1}-pm_n \mid \\
&=\mid q((x-1)x^{n+1}-m_{n+1})-p((x-1)x^n-m_n) \mid \\
&\leqslant q \mid (x-1)x^{n+1}-m_{n+1} \mid + p \mid (x-1)x^n-m_n \mid \\
&< \frac{q}{p+q}+\frac{p}{p+q}=1.
\end{aligned}$$

Therefore $m_{n+1}=\frac{p}{q}m_n$, $n \geqslant N$.

It follows that $m_{n+k}=\frac{p^k}{q^k}m_n$, $k \in \mathbf{N}^*$, $n \geqslant N$. Now let $k \to +\infty$ and n large enough such that

$$m_n > (x-1)x^n - 1 > 0.$$

We can conclude that $q = 1$. So x (> 1) is an integer.

3 There are 63 points on a circle C with radius 10. Let S be the number of triangles whose sides are longer than 9 and whose vertices are chosen from the 63 points. Find the maximum value of S.

Solution Let O be the center of circle C, a_n is the length of a regular n-gon $A_1A_2\cdots A_n$ inscribed in $\odot O$. Then $a_6 = 10 > 9$, $a_7 < 10 \times \frac{2\pi}{7} < 10 \times \frac{2 \times 3.15}{7} < 9$.

(1) Let $A_1A_2\cdots A_6$ be a regular 6-gon inscribed in $\odot O$, then $A_iA_{i+1} = a_6 > 9$. So we can choose a point B_i in $\widehat{A_iA_{i+1}}$ such that $B_iA_{i+1} > 9$. Then $\angle B_iOA_{i+1} > \frac{2\pi}{7}$ ($A_7 = A_1$), and

$$\angle A_iOB_i = \angle A_iOA_{i+1} - \angle B_iOA_{i+1} < \frac{2\pi}{6} - \frac{2\pi}{7} < \frac{2\pi}{7}.$$

It follows that $A_iB_i < 9$ ($i = 1, 2, \cdots, 6$).

In each of $\widehat{A_1B_1}$, $\widehat{A_2B_2}$, $\widehat{A_3B_3}$, choose 11 points, and in each of $\widehat{A_4B_4}$, $\widehat{A_5B_5}$, $\widehat{A_6B_6}$, choose 10 points. We obtain a set M which has 63 points on the circle C. It is easy to see for M the value of S is S_0, where

$$\begin{aligned} S_0 &= \binom{3}{3} \times 11^3 + \binom{3}{2} \cdot \binom{3}{1} \times 11^2 \times 10 \\ &\quad + \binom{3}{1} \cdot \binom{3}{2} \times 11 \times 10^2 + \binom{3}{3} \times 10^3 \\ &= 23\,121. \end{aligned}$$

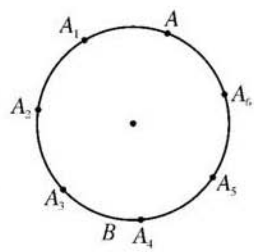
So the maximum value of S is not less than S_0.

(2) We prove that the maximum is S_0. We need three

lemmas.

Lemma 1 *For P on circle C, we call the arc $\widehat{APB}$ "an arc of P", if P is the midpoint of arc $\widehat{APB}$, and $\angle AOB = \frac{4\pi}{7}$. Now for every given n points on circle C, there is a point P, such that there are $\left[\frac{n+5}{6}\right]$ points of the given n points on the "arc of P".*

Proof: Let A be one of the given n points, and an "arc of A" be $\widehat{A_1AA_6}$. Now suppose A_2, A_3, $\cdots$, A_5 are on the arc $\widehat{A_1A_6}$ (not including A), and $A_1A_2 = A_2A_3 = \cdots = A_5A_6$ (see the figure). So $\angle A_iOA_{i+1} = \frac{2\pi}{7}$, $i = 1, 2, \cdots, 5$.



If there is a point P_i (of the given n points) on $\widehat{A_iA_{i+1}}$, then all the points (of the given n points) on $\widehat{A_iA_{i+1}}$ are on "an arc of P_i". So the given n points are on 6 "arcs of P_i" (including "arc of A"). This shows that there are $\left[\frac{n-1}{6}\right]+1 = \left[\frac{n+5}{6}\right]$ points of the given points on an "arc of P", where P is one of the given n points.

Lemma 2 *Take the arc $\widehat{A_1BA_6}$ arbitrary on the circle C with radius 10, where $\widehat{A_1BA_6}$ is $\frac{5}{7}$ of the perimeter. Then take any $5m + r$ points on the arc $\widehat{A_1BA_6}$ (m, r are non-negative integers and $0 \leqslant r < 5$). Prove that number of lines from the given points whose lengths are more than 9 is at most*

$$10m^2 + 4rm + \frac{1}{2}r(r-1).$$

Proof: Divide $\widehat{A_1BA_6}$ in five equal parts, where the corresponding points are A_2, A_3, A_4, A_5 (see the figure), then the length of $\widehat{A_iA_{i+1}}$ is exactly $\frac{1}{7}$ of the perimeter ($i = 1, 2, 3, 4, 5$), and the distance of any two points is not more than $a_7 < 9$. Suppose there are m_i given points on the arc $\widehat{A_iA_{i+1}}$, then the number of lines from the given points whose lengths are more than 9 is at most

$$l = \sum_{1 \leqslant i < j \leqslant 5} m_i m_j, \qquad ①$$

where

$$m_1 + m_2 + \cdots + m_5 = 5m + r. \qquad ②$$

Since there are finitely many non-negative integer groups (m_1, m_2, m_3, m_4, m_5), the maximum value of l exists. Now we prove that when the maximum is attained the inequality

$$|m_i - m_j| \leqslant 1, \ (1 \leqslant i < j \leqslant 5),$$

must hold.

In fact, if there exist i, j ($1 \leqslant i < j \leqslant 5$) such that $|m_i - m_j| \geqslant 2$ when the maximum is attained, we can suppose $m_1 - m_2 \geqslant 2$. Then let

$$m_1' = m_1 - 1, \ m_2' = m_2 + 1, \ m' = m,$$

and the corresponding integer is l', we will have

$$m_1' + m_2' = m_1 + m_2,$$

$$m_1' + m_2' + m_3' + m_4' + m_5' = m_1 + m_2 + m_3 + m_4 + m_5,$$

$$\begin{aligned} l' - l &= (m_1'm_2' - m_1m_2) + [(m_1' + m_2') \\ &\quad - (m_1 + m_2)](m_3 + m_4 + m_5) \\ &= m_1 - m_2 - 1 \geqslant 1. \end{aligned}$$

Contradiction!

Therefore, when l reaches the maximum value, the number of $m+1$ is r and the number of m is $5-r$. Thus, the number of lines from the given points whose lengths are more than 9 are at most

$$\binom{r}{2}(m+1)^2+\binom{r}{1}\binom{5-r}{1}(m+1)m+\binom{5-r}{2}m^2$$

$$=10m^2+4rm+\frac{1}{2}r(r-1).$$

Lemma 3 *Take arbitrary n points on the circle C with radius 10 to form set M, where $n=6m+r$ (m, r are non-negative integers, $0\leqslant r<6$). Assume that there are S_n triangles whose vertices are from M and each side is longer than 9. Prove*

$$S_n\leqslant 20m^3+10rm^2+2r(r-1)m+\frac{1}{6}r(r-1)(r-2).$$

Proof: We shall prove by mathematical induction.

When $n=1, 2$, $S_n=0$. It is true.

Suppose when $n=k$, it is true and set $k=6m+r$ (m, r are non-negative integers, $0\leqslant r<6$). Then

$$S_k\leqslant 20m^3+10rm^2+2r(r-1)m+\frac{1}{6}r(r-1)(r-2).$$

From Lemma 1, when $n=k+1$, the $k+1$ given points must include the point P, where at least $\left[\frac{k+1+5}{6}\right]=m+1$ given points are in the $\frac{2}{7}$ arc $\widehat{A_1PA_6}$. And the distances of such points to P are $\leqslant PA_1=PA_6=a_7<9$. Hence, there are at most $(k+1)-(m+1)=5m+r$ given points whose distances to P are more than 9, and such points are all in the other

$\frac{5}{7}$arc $\widehat{A_1PA_6}$ without P. From Lemma 2, the lines from such points whose lengths are more than 9 are at most

$$10m^2 + 4rm + \frac{1}{2}r(r-1).$$

(From Lemma 2, when $r = 5$, it is $10(m+1)^2$, which is also true.) Thus, the number of triangles whose vertex is P and each side is larger than 9 is not more than

$$S_p = 10m^2 + 4rm + \frac{1}{2}r(r-1).$$

Without P, there are $k = 6m + r$ given points. Let there be S_k triangles whose vertices are from the k points and each side is larger than 9, then using mathematical induction, we get

$$S_k \leqslant 20m^3 + 10rm^2 + 2r(r-1)m + \frac{1}{6}r(r-1)(r-2).$$

Furthermore

$$\begin{aligned} S_{k+1} &= S_k + S_p \\ &\leqslant 20m^3 + 10rm^2 + 2r(r-1)m + \frac{1}{6}r(r-1)(r-2) \\ &\quad + 10m^2 + 4rm + \frac{1}{2}r(r-1) \\ &= 20m^3 + 10(r+1)m^2 + 2r(r+1)m \\ &\quad + \frac{1}{6}r(r-1)(r+1), \end{aligned}$$

which means the case $n = k + 1 = 6m + (r+1)$ is also true.

On the other hand, when $r = 5$, then $m = k + 1 = 6(m+1)$ and S_{k+1} can be simplified to $S_{k+1} = 20(m+1)^3$, which is also true.

Therefore, we have proved Lemma 3.

Now considering the original problem, we have

$n=63=6\times 10+3$. It follows from Lemma 3,

$$S\leqslant 20\times 10^3+10\times 10^2+2\times 3\times 2\times 10+\frac{1}{6}\times 3\times 2\times 1$$
$$=23\,121.$$

Thus, $S_{\max}=23\,121$.

Second Day

0800 - 1230 April 1, 2007

4 Find all functions $f:\mathbf{Q}^+\to\mathbf{Q}^+$ such that

$$f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}, \quad ①$$

Where $\mathbf{Q}^+=\{q\mid q \text{ is a positive rational number}\}$.

Solution (1) Prove that $f(1)=1$.

Put $y=1$ in ①, and write $f(1)=a$. Then

$$f(x)+a+2xf(x)=\frac{f(x)}{f(x+1)}.$$

Thus

$$f(x+1)=\frac{f(x)}{(1+2x)f(x)+a}. \quad ②$$

Hence

$$f(2)=\frac{a}{4a}=\frac{1}{4},$$

$$f(3)=\frac{\frac{1}{4}}{\frac{5}{4}+a}=\frac{1}{5+4a},$$

$$f(4)=\frac{1}{7+5a+4a^2}.$$

On the other hand, we put $x = y = 2$ in ①, then

$$2f(2) + 8f(4) = \frac{f(4)}{f(4)} = 1.$$

By ②, we have

$$\frac{1}{2} + \frac{8}{7 + 5a + 4a^2} = 1.$$

Solving the equation, we have $a = 1$, i.e. $f(1) = 1$.

(2) Prove that

$$f(x+n) = \frac{f(x)}{(n^2 + 2nx)f(x) + 1}, \ n = 1, 2, \cdots. \qquad ③$$

Firstly, according to ② we know that ③ is true for $n = 1$. Now suppose ③ is true for $n = k$. Then

$$\begin{aligned} f(x+k+1) &= \frac{f(x+k)}{(1+2(x+k))f(x+k)+1} \\ &= \left(\frac{f(x)}{(k^2+2kx)f(x)+1}\right) \Big/ \left(\frac{(1+2(x+k))f(x)}{(k^2+2kx)f(x)+1} + 1\right) \\ &= \frac{f(x)}{((k+1)^2 + 2(k+1)x)f(x) + 1}. \end{aligned}$$

Hence the result follows by induction.

From ③ we have

$$f(n+1) = \frac{f(1)}{(n^2+2n)f(1)+1} = \frac{1}{(n+1)^2},$$

so $f(n) = \dfrac{1}{n^2}$, $n = 1, 2, \cdots$.

(3) Prove that

$$f\left(\frac{1}{n}\right) = n^2 = \frac{1}{\left(\frac{1}{n}\right)^2}, \ n = 1, 2, \cdots. \qquad ④$$

In fact, by letting $x = \frac{1}{n}$ in ③, we have

$$f\left(n+\frac{1}{n}\right)=\frac{f\left(\frac{1}{n}\right)}{(n^2+2)f\left(\frac{1}{n}\right)+1},$$

and by setting $y = \frac{1}{x}$ in ①, we have

$$f(x)+f\left(\frac{1}{x}\right)+2=\frac{1}{f\left(x+\frac{1}{x}\right)}.$$

So

$$f(n)+f\left(\frac{1}{n}\right)+2=\frac{1}{f\left(n+\frac{1}{n}\right)}=n^2+2+\frac{1}{f\left(\frac{1}{n}\right)}.$$

Consequently, $f(n) = \frac{1}{n^2}$ implies $f\left(\frac{1}{n}\right)=n^2$.

(4) Prove that if $q = \frac{n}{m}$, $\gcd(m, n) = 1$, $m, n \in \mathbf{N}^*$, then $f(q) = \frac{1}{q^2}$.

For $m, n \in \mathbf{N}^*$, $\gcd(m, n) = 1$, put $x = n$, $y = \frac{1}{m}$ in ①, we have

$$f\left(\frac{1}{m}\right)+f(n)+\frac{2n}{m}f\left(\frac{n}{m}\right)=\frac{f\left(\frac{n}{m}\right)}{f\left(n+\frac{1}{m}\right)}.$$

Put $x = \frac{1}{m}$ in ③, we get

$$f\left(n+\frac{1}{m}\right)=\frac{f\left(\frac{1}{m}\right)}{\left(n^2+\frac{2n}{m}\right)f\left(\frac{1}{m}\right)+1}=\frac{1}{n^2+\frac{2n}{m}+\frac{1}{m^2}}.$$

So

$$\frac{1}{n^2}+m^2+\frac{2n}{m}f\left(\frac{n}{m}\right)=\left(n+\frac{1}{m}\right)^2 f\left(\frac{n}{m}\right).$$

Now we have

$$f(q)=f\left(\frac{n}{m}\right)=\frac{\frac{1}{n^2}+m^2}{n^2+\frac{1}{m^2}}=\left(\frac{m}{n}\right)^2=\frac{1}{q^2}.$$

Finally, it is easy to verify that $f(x)=\frac{1}{x^2}$ satisfies the condition. So $f(x)=\frac{1}{x^2}$ is the answer.

5 Let $x_1, \cdots, x_n (n \geqslant 2)$ be real numbers such that

$$A=\left|\sum_{i=1}^{n} x_i\right| \neq 0$$

and

$$B=\max_{1 \leqslant i<j \leqslant n} |x_i-x_j| \neq 0.$$

Prove that for every n vectors $\alpha_1, \cdots, \alpha_n$ on the plane, there exists a permutation $(k_1, k_2, \cdots, k_n)$ of $(1, 2, \cdots, n)$ such that

$$\left|\sum_{i=1}^{n} x_{k_i}\alpha_i\right| \geqslant \frac{AB}{2A+B}\max_{1 \leqslant i \leqslant n} |\alpha_i|.$$

Proof Let $|\alpha_k|=\max_{1 \leqslant i \leqslant n}|\alpha_i|$. It is sufficient to prove that

$$\max_{(k_1, \cdots, k_n) \in S_n}\left|\sum_{i=1}^{n} x_{k_i}\alpha_i\right| \geqslant \frac{AB}{2A+B}|\alpha_k|,$$

where S_n is the set of all permutations of $(1, 2, \cdots, n)$.

Without loss of generality, assume

$$|x_n - x_1| = \max_{1\leqslant i<j\leqslant n} |x_j - x_i| = B,$$

$$|\alpha_n - \alpha_1| = \max_{1\leqslant i<j\leqslant n} |\alpha_j - \alpha_i|.$$

For the two vectors

$$\beta_1 = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} + x_n\alpha_n,$$

$$\beta_2 = x_n\alpha_1 + x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} + x_1\alpha_n,$$

we have

$$\begin{aligned}
\max_{(k_1, \cdots, k_n)\in S_n} \left| \sum_{i=1}^{n} x_{k_i}\alpha_i \right| &\geqslant \max\{|\beta_1|, |\beta_2|\} \\
&\geqslant \frac{1}{2}(|\beta_1| + |\beta_2|) \\
&\geqslant \frac{1}{2}|\beta_1 - \beta_2| \\
&= \frac{1}{2}|x_1\alpha_n + x_n\alpha_1 - x_1\alpha_1 - x_n\alpha_n| \\
&= \frac{1}{2}|x_1 - x_n| \cdot |\alpha_1 - \alpha_n| \\
&= \frac{1}{2}B|\alpha_n - \alpha_1|. \qquad ①
\end{aligned}$$

Now suppose $|\alpha_n - \alpha_1| = x|\alpha_k|$. Using the Triangle Inequality, we obtain $0 \leqslant x \leqslant 2$. So ① becomes

$$\max_{(k_1, \cdots, k_n)\in S_n} \left| \sum_{i=1}^{n} x_{k_i}\alpha_i \right| \geqslant \frac{1}{2}Bx|\alpha_k|. \qquad ②$$

On the other hand, consider the vectors

$$\gamma_1 = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} + x_n\alpha_n$$

$$\gamma_2 = x_2\alpha_1 + x_3\alpha_2 + \cdots + x_n\alpha_{n-1} + x_1\alpha_n$$

$$\cdots\cdots$$

$$\gamma_n = x_n\alpha_1 + x_1\alpha_2 + \cdots + x_{n-2}\alpha_{n-1} + x_{n-1}\alpha_n.$$

Then we have

$$\begin{aligned}
\max_{(k_1, \cdots, k_n)\in S_n} \left| \sum_{i=1}^{n} x_{k_i}\alpha_i \right| &\geqslant \max_{1\leqslant i\leqslant n} |\gamma_i| \\
&\geqslant \frac{1}{n}(|\gamma_1| + \cdots + |\gamma_n|) \\
&\geqslant \frac{1}{n}|\gamma_1 + \cdots + \gamma_n| \\
&= \frac{A}{n}|\alpha_1 + \cdots + \alpha_n| \\
&= \frac{A}{n}\left| n\alpha_k - \sum_{j\neq k}(\alpha_k - \alpha_j) \right| \\
&\geqslant \frac{A}{n}\left(n|\alpha_k| - \sum_{j\neq k}|\alpha_k - \alpha_j|\right) \\
&\geqslant \frac{A}{n}(n|\alpha_k| - (n-1)|\alpha_n - \alpha_1|) \\
&= \frac{A}{n}(n|\alpha_k| - (n-1)x|\alpha_k|) \\
&= A\left(1 - \frac{n-1}{n}x\right)|\alpha_k|. \qquad ③
\end{aligned}$$

From ② and ③, it follows that

$$\begin{aligned}
\max_{(k_1, \cdots, k_n)\in S_n} \left| \sum_{i=1}^{n} x_{k_i}\alpha_i \right| &\geqslant \max\left\{\frac{Bx}{2}, A\left(1 - \frac{n-1}{n}x\right)\right\}|\alpha_k| \\
&\geqslant \frac{\frac{Bx}{2}\cdot A\cdot\frac{n-1}{n} + A\left(1 - \frac{n-1}{n}x\right)\cdot\frac{B}{2}}{A\cdot\frac{n-1}{n} + \frac{B}{2}}|\alpha_k| \\
&= \frac{AB}{2A + B - \frac{2A}{n}}|\alpha_k| \\
&\geqslant \frac{AB}{2A+B}|\alpha_k|.
\end{aligned}$$

6 Let n be a positive integer, set $A \subseteq \{1, 2, \cdots, n\}$, and for every a, $b \in A$, $\operatorname{lcm}(a, b) \leqslant n$. Prove that

$$|A| \leqslant 1.9\sqrt{n} + 5.$$

Proof For $a \in (\sqrt{n}, \sqrt{2n}]$, $\operatorname{lcm}(a, a+1) = a(a+1) > n$, so $|A \cap (\sqrt{n}, \sqrt{2n}]| \leqslant \frac{1}{2}(\sqrt{2}-1)\sqrt{n}+1$.

For $a \in (\sqrt{2n}, \sqrt{3n}]$, we have

$$\operatorname{lcm}(a, a+1) = a(a+1) > n,$$

$$\operatorname{lcm}(a+1, a+2) = (a+1)(a+2) > n,$$

$$\operatorname{lcm}(a, a+2) \geqslant \frac{1}{2}a(a+2) > n.$$

So

$$|A \cap (\sqrt{2n}, \sqrt{3n}]| \leqslant \frac{1}{3}(\sqrt{3}-\sqrt{2})\sqrt{n}+1.$$

Similarly

$$|A \cap (\sqrt{3n}, 2\sqrt{n}]| \leqslant \frac{1}{4}(\sqrt{4}-\sqrt{3}) \cdot \sqrt{n}+1.$$

Hence

$$\begin{aligned}|A \cap [1, 2\sqrt{n}]| &\leqslant \sqrt{n} + \frac{1}{2}(\sqrt{2}-1)\sqrt{n} + \frac{1}{3}(\sqrt{3}-\sqrt{2})\sqrt{n} \\ &\quad + \frac{1}{4}(\sqrt{4}-\sqrt{3})\sqrt{n}+3 \\ &= \left(1+\frac{\sqrt{2}}{6}+\frac{\sqrt{3}}{12}\right)\sqrt{n}+3.\end{aligned}$$

Let $k \in \mathbf{N}^*$, suppose $a, b \in \left(\frac{n}{k+1}, \frac{n}{k}\right)$, $a > b$, and $\operatorname{lcm}(a, b) = as = bt$, where $s, t \in \mathbf{N}^*$. Then

$$\frac{a}{(a, b)}s = \frac{b}{(a, b)}t.$$

Since $\gcd\left(\frac{a}{(a, b)}, \frac{b}{(a, b)}\right) = 1$, so $\frac{b}{(a, b)} \Big| s$. It follows that

$$\begin{aligned}\operatorname{lcm}(a, b) &= as \geqslant \frac{ab}{(a, b)} \geqslant \frac{ab}{a-b} \\ &= b + \frac{b^2}{a-b} > \frac{n}{k+1} + \frac{\left(\frac{n}{k+1}\right)^2}{\frac{n}{k} - \frac{n}{k+1}} \\ &= n.\end{aligned}$$

Therefore, $\left|A \cap \left(\frac{n}{k+1}, \frac{n}{k}\right]\right| \leqslant 1$.

Suppose $T \in \mathbf{N}^*$ such that $\frac{n}{T+1} \leqslant 2\sqrt{n} < \frac{n}{T}$. Then

$$\begin{aligned}|A \cap (2\sqrt{n}, n]| &\leqslant \sum_{k=1}^{T} \left|A \cap \left(\frac{n}{k+1}, \frac{n}{k}\right]\right| \\ &\leqslant T < \frac{1}{2}\sqrt{n}.\end{aligned}$$

By the above arguments, we arrive at

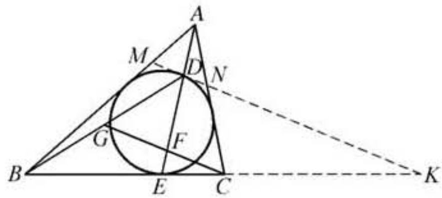
$$|A| \leqslant \left(\frac{3}{2} + \frac{1}{6}\sqrt{2} + \frac{1}{12}\sqrt{3}\right)\sqrt{n} + 3 < 1.9\sqrt{n} + 5.$$

2008 (Suzhou, Jiangsu)

First Day

0800 - 1230 Mar 31, 2008

1. In triangle ABC, we have $AB > AC$. The incircle ω touches BC at E, and AE intersects ω at D. Choose a point F on AE (F is different from E), such that $CE = CF$. Let G be the intersection point of CF and BD. Prove that $CF = FG$.



Proof Referring to the figure, draw a line from D, tangent to ω, and the line intersects AB, AC, BC at points M, N, K respectively.

Since

$$\angle KDE = \angle AEK = \angle EFC,$$

we know $MK \parallel CG$.

By *Newton's* theorem, the lines BN, CM, DE are concurrent.

By *Ceva's* theorem, we have

$$\frac{BE}{EC} \cdot \frac{CN}{NA} \cdot \frac{AM}{MB} = 1. \qquad ①$$

From *Menelaus'* theorem,

$$\frac{BK}{KC} \cdot \frac{CN}{NA} \cdot \frac{AM}{MB} = 1. \qquad ②$$

① ÷ ②, we have

$$BE \cdot KC = EC \cdot BK,$$

thus

$$BC \cdot KE = 2EB \cdot CK. \qquad ③$$

Using *Menelaus'* theorem and ③, we get

$$1 = \frac{CB}{BE} \cdot \frac{ED}{DF} \cdot \frac{FG}{GC} = \frac{CB}{BE} \cdot \frac{EK}{CK} \cdot \frac{FG}{GC} = \frac{2FG}{GC}.$$

So $CF = GF$.

2 The sequence $\{x_n\}$ is defined by $x_1 = 2$, $x_2 = 12$, $x_{n+2} =$

$6x_{n+1} - x_n$, $n = 1, 2, \cdots$. Let p be an odd prime number, and q be a prime number such that $q \mid x_p$. Prove that if $q \neq 2$, then $q \geqslant 2p - 1$.

Proof It is easy to see

$$x_n = \frac{1}{2\sqrt{2}}((3+2\sqrt{2})^n - (3-2\sqrt{2})^n),\ n = 1, 2, \cdots.$$

Let a_n, b_n be positive integers and $a_n + b_n\sqrt{2} = (3+2\sqrt{2})^n$. Then

$$a_n - b_n\sqrt{2} = (3-2\sqrt{2})^n,$$

so $x_n = b_n$, $a_n^2 - 2b_n^2 = 1$, $n = 1, 2, \cdots$.

Suppose $q \neq 2$. Since $q \mid x_p$, thus $q \mid b_p$, so there exists a term in $\{b_n\}$ which is divisible by q. Let d be the least number such that $q \mid b_d$. We have the following lemma.

Lemma *For any positive integer n, $q \mid b_n$ if and only if $d \mid n$.*

Proof: For $a, b, c, d \in \mathbf{Z}$, denote $a + b\sqrt{2} \equiv c + d\sqrt{2} \pmod q$ as $a \equiv c \pmod q$ and $b \equiv d \pmod q$.

If $d \mid n$, write $n = du$, then

$$a_n + b_n\sqrt{2} = (3+2\sqrt{2})^{du} \equiv a_d^u \pmod q,$$

so $b_n \equiv 0 \pmod q$.

On the other hand, if $q \mid b_n$, write $n = du + r$, $0 \leqslant r < d$. Suppose $r \geqslant 1$, from

$$\begin{aligned} a_n &= (3+2\sqrt{2})^n = (3+2\sqrt{2})^{du} \cdot (3+2\sqrt{2})^r \\ &\equiv a_d^u (a_r + b_r\sqrt{2}) \pmod q, \end{aligned}$$

we have

$$a_d^u b_r \equiv 0 \pmod q. \qquad ①$$

But $a_d^2 - 2b_d^2 = 1$, and $q \mid b_d$; so $q \nmid a_d^2$. Since q is a prime, therefore $q \nmid a_d$, and $(q, a_d^u) = 1$, From ① we have $q \mid b_r$, it contradicts the definition of d. So $r = 0$, and the lemma is proven.

Now, as q is a prime, So $q \mid \binom{q}{i}$, $i = 1, 2, \cdots, q-1$. Using the *Fermat's* little theorem, we have

$$3^q \equiv 3 \pmod q,\ 2^q \equiv 2 \pmod q.$$

As $q \neq 2$, so $2^{\frac{q-1}{2}} \equiv \pm 1 \pmod q$, we get

$$\begin{aligned}(3+2\sqrt{2})^q &= \sum_{i=0}^{q}\binom{q}{i}\cdot 3^{q-i}(2\sqrt{2})^i \\ &\equiv 3^q + (2\sqrt{2})^q \\ &= 3^q + 2^q \cdot 2^{\frac{q-1}{2}}\sqrt{2} \\ &\equiv 3 \pm 2\sqrt{2} \pmod q.\end{aligned}$$

By the same argument, we have

$$(3+2\sqrt{2})^{q^2} \equiv (3 \pm 2\sqrt{2})^q \equiv 3 + 2\sqrt{2} \pmod q.$$

So

$$(a_{q^2-1} + \sqrt{2}b_{q^2-1})(3+2\sqrt{2}) \equiv 3 + 2\sqrt{2} \pmod q.$$

Thus,

$$\begin{cases} 3a_{q^2-1} + 4b_{q^2-1} \equiv 3 \pmod q, \\ 2a_{q^2-1} + 3b_{q^2-1} \equiv 2 \pmod q. \end{cases}$$

We know that $q \mid b_{q^2-1}$.

Since $q \mid b_p$, from the lemma, we have $d \mid p$. So $d \in \{1, p\}$, and if $d = 1$, then $q \mid b_1 = 2$, contradiction! So $d = p$, hence $q \mid b_{q^2-1}$. So $p \mid q^2 - 1$, thus $p \mid q-1$ or $p \mid q+1$. Since $q-1$ and $q+1$ are even, so $q \geqslant 2p - 1$.

3 Every positive integer is colored by blue or red. Prove that there is a sequence $\{a_n\}$ which has infinite terms, and $a_1 < a_2 < \cdots$ are positive integers, such that a_1, $\frac{a_1+a_2}{2}$, a_2, $\frac{a_2+a_3}{2}$, a_3, $\cdots$ is a positive integer sequence with the same color.

Proof We need three lemmas. Firstly, define $\mathbf{N}^*$ as the set of all the positive integers.

Lemma 1 *If there is an arithmetic progression having infinite positive integer terms with the same color, then the conclusion holds.*

Proof: Let $c_1 < c_2 < \cdots < c_n < \cdots$ be a red arithmetic progression of positive integers, we can set $a_i = c_{2i-1}$ ($i = 1, 2, 3, \cdots$) to obtain a sequence such that the condition holds.

Lemma 2 *If for any $i \in \mathbf{N}^*$, there exists a positive integer j, such that i, $\frac{i+j}{2}$, j are of the same color, then the conclusion holds.*

Proof: Let $a_1 = 1$, and a_1 be red. Since there exists $k \in \mathbf{N}^*$ such that a_1, $\frac{a_1+k}{2}$, k have the same color, so we can set $a_2 = k$. In the same way, we can have a sequence of red numbers satisfying

$$a_1 < \frac{a_1+a_2}{2} < a_2 < \frac{a_2+a_3}{2} < a_3 < \cdots.$$

Lemma 3 *If there is no arithmetic progression satisfying the condition of Lemma 1, and there exists $i_0 \in \mathbf{N}^*$, such that for every $j \in \mathbf{N}^*$, i_0, $\frac{i_0+j}{2}$, j have different colors, then the* *conclusion holds.*

Proof: We can suppose $i_0 = 1$, otherwise using $N' = \{ni_0 \mid n \in \mathbf{N}^*\}$ in place of $\mathbf{N}^*$, will yield the same result.

Let $i_0 = 1$ be a red number. Then we have the following condition: For every $k \in \mathbf{N}^*$, $k \geqslant 2$, k and $2k - 1$ cannot be both red. ①

Since there is no arithmetic progression having infinite terms with the same color, therefore there are infinite terms of blue colored numbers of different parities in $\mathbf{N}^*$. We prove there is an infinite sequence of odd numbers of blue color in $\mathbf{N}^*$.

Let a_1 be a blue odd number, and suppose odd numbers $a_1 < a_2 < \cdots < a_n$ satisfy that

$$a_1 < \frac{a_1 + a_2}{2} < a_2 < \cdots < a_{n-1} < \frac{a_{n-1} + a_n}{2} < a_n$$

are all blue. Now we prove there exists an odd number $a_{n+1} \in \mathbf{N}^*$, such that $\frac{a_n + a_{n+1}}{2}$ and a_{n+1} are blue. ②

(1) If for every $i \in \mathbf{N}^*$, the numbers $a_n + i$, $a_n + 2i$ have different colors, and there is no a_{n+1} satisfying ②, then for $a_{n+1} > a_n$, numbers $\frac{a_n + a_{n+1}}{2}$ and a_{n+1} cannot be both blue. ③

Since there is no arithmetic progression with infinite terms of the same color, there must exist infinitely many red and blue numbers in $\mathbf{N}^*$. Let $i \in \mathbf{N}^*$ such that $a_n + i$ is red, then $a_n + 2i$ is blue. Write $a_n = 2k + 1$, we know $2k + 1$ is blue, $2k + i + 1$ is red, $2k + 2i + 1$ is blue. Using ①, we know that $4k + 2i + 1 (= 2(2k + i + 1) - 1)$ is blue. Similarly from ③, $3k + i +$

$1\left(=\dfrac{(2k+1)+(4k+2i+1)}{2}\right)$ is red. From ① we get $6k+2i+1(=2(3k+i+1)-1)$ to be blue, and so on. Consequently, we have an arithmetic progression $\{2nk+2i+1\}_{i=1}^{+\infty}$ of blue numbers, contradiction! Hence there must be a number a_{n+1} satisfying ②.

(2) Suppose there is $i\in\mathbf{N}^*$, such that a_n+i and a_n+2i have the same color. Let $a_n=2k+1$.

Firstly, if a_n+i and a_n+2i are blue, then $a_{n+1}=a_n+2i$, satisfying ②.

Secondly, if a_n+i and a_n+2i are red, then from ① we have $4k+2i+1\ (=2(2k+i+1)-1)$ and $4k+4i+1\ (=2(2k+2i+1)-1)$ being blue. So if $3k+2i+1\ \left(=\dfrac{(2k+1)+(4k+4i+1)}{2}\right)$ are blue, then $a_{n+1}=4k+4i+1$, hence satisfying ②, otherwise, the number $6k+4i+1\ (=2(3k+2i+1)-1)$ is blue, then $a_{n+1}=6k+4i+1$, satisfying ②. Hence proving Lemma 3.

Therefore combining Lemmas 1, 2 and 3, we are done.

Second Day

0800 - 1230 April, 2008

4 Let $n\in\mathbf{N}^*$, $n\geqslant 4$, and $G_n=\{1, 2, \cdots, n\}$. Prove that there is a permutation $P_1, P_2, \cdots, P_{2^n-n-1}$, where $P_i\subseteq G_n$, $|P_i|\geqslant 2$, $i=1, 2, \cdots, 2^n-n-1$, such that

$$|P_i\cap P_{i+1}|=2,\ i=1, 2, \cdots, 2^n-n-2. \qquad ①$$

Proof Our proof will require the following lemma.

Lemma *For $n\in\mathbf{N}^*$, $n\geqslant 3$, there is a permutation Q_1,*

Q_2, $\cdots$, Q_{2^n-1}, *where* $Q_i \subseteq G_n$, $|Q_i| \geqslant 1$, $1 \leqslant i \leqslant 2^n - 1$, *such that*

$$Q_1 = \{1\},\ |Q_i \cap Q_{i+1}| = 1,\ 1 \leqslant i \leqslant 2^n - 2,\ Q_{2^n-1} = G_n. \qquad ②$$

Firstly, for $n = 3$, the permutation

$$\{1\},\ \{1, 2\},\ \{2\},\ \{2, 3\},\ \{1, 3\},\ \{3\},\ \{1, 2, 3\}$$

satisfies the given condition.

Secondly, suppose the lemma is true for n. Let Q_1, Q_2, $\cdots$, Q_{2^n-1} satisfy the conditions in the lemma. Then construct the following sequence:

Q_1, Q_{2^n-1}, Q_{2^n-2}, $Q_{2^n-2} \cup \{n+1\}$, Q_{2^n-3}, $Q_{2^n-4} \cup \{n+1\}$, Q_{2^n-5}, $\cdots$, Q_3, $Q_2 \cup \{n+1\}$, $\{n+1\}$, $Q_1 \cup \{n+1\}$, Q_2, $Q_3 \cup \{n+1\}$, Q_4, $\cdots$, Q_{2^n-2}, $Q_{2^n-1} \cup \{n+1\}$.

It is easy to check that ③ satisfies the lemma for $n+1$.

Back to the problem, we prove that for $n \in \mathbf{N}^*$, $n \geqslant 4$, there is a permutation satisfying ① and $P_{2^n-n-1} = \{1, n\}$.

When $n = 4$, the permutation

$\{1, 3\}$, $\{1, 2, 3\}$, $\{2, 3\}$, $\{1, 2, 3, 4\}$, $\{1, 2\}$, $\{1, 2, 4\}$, $\{2, 4\}$, $\{2, 3, 4\}$, $\{3, 4\}$, $\{1, 3, 4\}$, $\{1, 4\}$

also satisfies the given condition.

Now suppose the permutation P_1, P_2, $\cdots$, P_{2^n-n-1} satisfies ①, and $P_{2^n-n-1} = \{1, n\}$. Using the lemma, let the permutation Q_1, Q_2, $\cdots$, Q_{2^n-1} satisfies ②. Then for $n+1$, the following permutation

P_1, P_2, $\cdots$, P_{2^n-n-1}, $Q_{2^n-1} \cup \{n+1\}$, $Q_{2^n-2} \cup \{n+1\}$, $\cdots$, $Q_1 \cup \{n+1\}$

satisfies ① and $P_{2^{n+1}-(n+1)-1} = Q_1 \cup \{n+1\} = \{1, n+1\}$.

So there is a suitable permutation for $n \geqslant 4$.

5 Let $m, n \in \mathbf{N}^*$, $m, n > 1$, a_{ij} $(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ be non-negative real numbers (not all zero). Find the maximum and minimum values of

$$f = \frac{n \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij})^2 + m \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij})^2}{(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij})^2 + mn \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2}.$$

Solution The maximum value of f is 1.

Firstly, we prove that $f \leqslant 1$. It suffices to show that

$$\begin{aligned} & n \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij})^2 + m \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij})^2 \\ \leqslant & (\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij})^2 + mn \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2, \end{aligned}$$

or

$$(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij})^2 + mn \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2 - n \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij})^2 - m \sum_{j=1}^{m} (\sum_{i=1}^{n} a_{ij})^2 \geqslant 0,$$

or

$$\sum_{\substack{1 \leqslant p < s \leqslant n \\ 1 \leqslant q < r \leqslant m}} (a_{pq} + a_{sr} - a_{pr} - a_{sq})^2 \geqslant 0.$$

So $f \leqslant 1$, and when all of a_{ij} are equal to 1, $f = 1$.

The minimum value of f is $\frac{m+n}{mn + \min\{m, n\}}$.

To prove $f \geqslant \frac{m+n}{mn + \min\{m, n\}}$, without loss of generality, we assume $n \leqslant m$. Hence it is sufficient to prove that

$$f \geqslant \frac{m+n}{mn+n}. \qquad ①$$

Let

$$S = \frac{n^2(m+1)}{m+n} \sum_{i=1}^{n} r_i^2 + \frac{mn(m+1)}{m+n} \sum_{j=1}^{m} c_j^2$$

$$-(\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij})^2-mn\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}^2,$$

where $r_i=\sum_{j=1}^{m}a_{ij}$, $1\leqslant i\leqslant n$, $c_j=\sum_{i=1}^{n}a_{ij}$, $i\leqslant j\leqslant m$.

Now ① $\Leftrightarrow S\geqslant 0$. Consider *Lagrange's* equation.

$$(\sum_{i=1}^{n}a_ib_i)^2=(\sum_{i=1}^{n}a_i^2)(\sum_{i=1}^{n}b_i^2)-\sum_{1\leqslant k<l\leqslant n}(a_kb_l-a_lb_k)^2.$$

Put $a_i=r_i$, $b_i=1$, $1\leqslant i\leqslant n$. Then

$$-(\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij})^2=-n\sum_{i=1}^{n}r_i^2+\sum_{1\leqslant k<l\leqslant n}(r_k-r_l)^2,$$

and

$$\begin{aligned}S&=\frac{mn(n-1)}{m+n}\sum_{i=1}^{n}r_i^2+\frac{mn(m+1)}{m+n}\sum_{j=1}^{m}c_j^2\\&\quad-mn\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}^2+\sum_{1\leqslant k<l\leqslant n}(r_k-r_l)^2\\&=\frac{mn(n-1)}{m+n}\sum_{j=1}^{m}\sum_{i=1}^{n}a_{ij}(r_i-a_{ij})^2\\&\quad+\frac{mn(m+1)}{m+n}\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}(c_j-a_{ij})^2\\&\quad+\sum_{1\leqslant k<l\leqslant n}(r_k-r_l)^2.\end{aligned}$$

Since $a_{ij}\geqslant 0$, $r_i\geqslant a_{ij}$, $c_j\geqslant a_{ij}$, so $S\geqslant 0$.

When $a_{11}=a_{22}=\cdots=a_{nn}=1$ and the other $a_{ij}=0$, the minimum value of f is $\frac{m+n}{mn+n}$.

With the above arguments, we conclude that the maximum value of f is 1 and the minimum value of f is $\frac{m+n}{mn+\min\{m,n\}}$.

6. Find the maximum positive number M such that for every $n \in \mathbf{N}^*$, there are positive numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ satisfying

(a) $\sum_{k=1}^{n} b_k = 1$, $2b_k \geqslant b_{k-1} + b_{k+1}$, $k = 2, 3, \cdots, n-1$,

(b) $a_k^2 \leqslant 1 + \sum_{i=1}^{k} a_i b_i$, $k = 1, 2, \cdots, n$,

(c) $a_n = M$.

Solution Firstly, we prove that

$$\max_{1 \leqslant k \leqslant n} a_k < 2, \text{ and } \max_{1 \leqslant k \leqslant n} b_k < \frac{2}{n-1}.$$

Let $L = \max_{1 \leqslant k \leqslant n} a_k$. From (b) and $\sum_{k=1}^{n} b_k = 1$, we get $L^2 \leqslant 1 + L$, so $L < 2$.

Let $b_m = \max_{1 \leqslant k \leqslant n} b_k$. Then by using $2b_k \geqslant b_{k-1} + b_{k+1}$, it is easy to see that

$$b_k \geqslant \begin{cases} \dfrac{(k-1)b_m + (m-k)b_1}{m-1}, & 1 \leqslant k \leqslant m, \\ \dfrac{(k-m)b_n + (n-k)b_m}{n-m}, & m \leqslant k \leqslant n. \end{cases}$$

Since $b_1 > 0$ and $b_n > 0$, so

$$b_k > \begin{cases} \dfrac{k-1}{m-1} b_m, & 1 \leqslant k \leqslant m, \\ \dfrac{n-k}{n-m} b_m, & m \leqslant k \leqslant n. \end{cases}$$

It follows that

$$\begin{aligned} 1 &= \sum_{k=1}^{n} b_k = \sum_{k=1}^{m} b_k + \sum_{k=m+1}^{n} b_k \\ &> \frac{1}{m-1}\Big(\sum_{k=1}^{m}(k-1)\Big)b_m + \frac{1}{n-m}\Big(\sum_{k=m+1}^{n}(n-k)\Big)b_m \end{aligned}$$

$$= \frac{m}{2}b_m + \frac{n-m-1}{2}b_m = \frac{n-1}{2}b_m.$$

So $b_m < \frac{2}{n-1}$, that is $\max\limits_{1 \leqslant k \leqslant n} b_k < \frac{2}{n-1}$.

Now let $f_0 = 1$, $f_k = 1 + \sum\limits_{i=1}^{k} a_i b_i$, $k = 1, 2, \cdots, n$. Then $f_k - f_{k-1} = a_k b_k$, and from (b) we have $a_k^2 \leqslant f_k$, i. e. $a_k \leqslant \sqrt{f_k}$, $k = 1, 2, \cdots, n$.

Since $\max\limits_{1 \leqslant k \leqslant n} a_k < 2$, so

$$f_k - f_{k-1} = a_k b_k \leqslant b_k \sqrt{f_k}$$

and

$$f_k - f_{k-1} < 2b_k.$$

Thus, for $1 \leqslant k \leqslant n$,

$$\begin{aligned}
\sqrt{f_k} - \sqrt{f_{k-1}} &< b_k \cdot \frac{\sqrt{f_k}}{\sqrt{f_k} + \sqrt{f_{k-1}}} \\
&= b_k\left(\frac{1}{2} + \frac{f_k - f_{k-1}}{2(\sqrt{f_k} + \sqrt{f_{k-1}})^2}\right) \\
&< b_k\left(\frac{1}{2} + \frac{2b_k}{2(\sqrt{f_k} + \sqrt{f_{k-1}})^2}\right) \\
&< b_k\left(\frac{1}{2} + \frac{b_k}{4}\right) \\
&< \left(\frac{1}{2} + \frac{1}{2(n-1)}\right)b_k.
\end{aligned}$$

Hence, summing from $k = 1$ to n,

$$\begin{aligned}
a_n \leqslant \sqrt{f_n} &< \sqrt{f_0} + \sum_{k=1}^{n}\left(\frac{1}{2} + \frac{1}{2(n-1)}\right)b_k \\
&= \frac{3}{2} + \frac{1}{2(n-1)}.
\end{aligned}$$

Let $n \to +\infty$, we obtain $a_n \leqslant \frac{3}{2}$.

When $a_k = 1 + \frac{k}{2n}$, $b_k = \frac{1}{n}$, $k = 1, 2, \cdots, n$, we have $a_k^2 = \left(1 + \frac{k}{2n}\right)^2 \leqslant 1 + \sum_{i=1}^{k} \frac{1}{n}\left(1 + \frac{i}{2n}\right)$. Hence the maximum value is $\frac{3}{2}$.

China Girls' Mathematical Olympiad

2006 (Urumqi, Xinjiang)

The 5th China Girls' Mathematical Olympiad was held on 7 – 11 August 2006 in Urumqi, Xinjiang, China, and was hosted by China Mathematical Olympiad Committee. The Competition Committee comprised: Zhu Huawei, Chen Yonggao, Su Chun, Li Weigu, Li Shenghong, Ye Zhonghao, Ji Chungang, Yuan Hanhui.

First Day

1500 - 1900 August 8, 2006

1 A function $f:(0, +\infty) \longrightarrow \mathbf{R}$ satisfies the following conditions:

(a) $f(a) = 1$ for a positive real number a,

(b) $$f(x)f(y) + f\left(\frac{a}{x}\right)f\left(\frac{a}{y}\right) = 2f(xy), \qquad ①$$

for any positive real number x, y. Prove that $f(x)$ is constant.

Proof Setting $x = y = 1$ in ① gives

$$f^2(1) + f^2(a) = 2f(1),$$
$$(f(1) - 1)^2 = 0,$$

so $f(1) = 1$.

Setting $y = 1$ in ① yields

$$f(x)f(1) + f\left(\frac{a}{x}\right)f(a) = 2f(x),$$
$$f(x) = f\left(\frac{a}{x}\right),\ x > 0. \qquad ②$$

Setting $y = \frac{a}{x}$ in ① yields

$$f(x)f\left(\frac{a}{x}\right) + f\left(\frac{a}{x}\right)f(x) = 2f(a),$$
$$f(x)f\left(\frac{a}{x}\right) = 1. \qquad ③$$

Combining ② and ③ gives $f^2(x) = 1$, $x > 0$.

Setting $x = y = \sqrt{t}$ in ① gives

$$f^2(\sqrt{t}) + f^2\left(\frac{a}{\sqrt{t}}\right) = 2f(t),$$

$$f(t) > 0.$$

So $f(x) = 1$, $x > 0$, as desired.

2 Let $ABCD$ be a convex quadrilateral. Let O be the intersection of AC and BD. Let O and M be the intersections of the circumcircle of $\triangle OAD$ with the circumcircle of $\triangle OBC$. Let T and S be the intersections of OM with the circumcircle of $\triangle OAB$ and $\triangle OCD$ respectively. Prove that M is the midpoint of TS.

Proof Since $\angle BTO = \angle BAO$ and $\angle BCO = \angle BMO$, $\triangle BTM$ and $\triangle BAC$ are similar. Hence,

$$\frac{TM}{AC} = \frac{BM}{BC}. \qquad ①$$

Similarly,

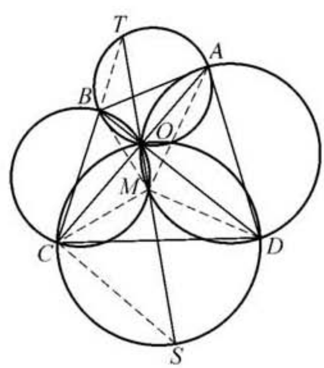
$$\triangle CMS \backsim \triangle CBD.$$

Hence,

$$\frac{MS}{BD} = \frac{CM}{BC}. \qquad ②$$

Dividing ① by ②, we have

$$\frac{TM}{MS} = \frac{BM}{CM} \cdot \frac{AC}{BD}. \qquad ③$$



Since $\angle MBD = \angle MCA$ and $\angle MDB = \angle MAC$, $\triangle MBD$ and $\triangle MCA$ are similar. Hence,

$$\frac{BM}{CM} = \frac{BD}{AC}. \qquad ④$$

Combining ④ and ③ yields $TM = MS$, as desired.

3 Prove that for $i = 1, 2, 3$, there exist infinitely many integers n satisfying the following condition: we can find i integers in $\{n, n+2, n+28\}$ that can be expressed as the sum of the cubes of three positive integers.

Proof We first prove a lemma.

Lemma *Let m be the remainder of the sum of the cubes of three positive integers when divided by 9, then $m \neq 4$ or 5.*

Proof: since any integer can be expressed as $3k$ or $3k \pm 1$ $(k \in \mathbf{Z})$, but

$$(3k)^3 = 9 \times 3k^3,$$
$$(3k \pm 1)^3 = 9 \times (3k^3 \pm 3k^2 + k) \pm 1,$$

as desired.

If $i = 1$, take $n = 3(3m-1)^3 - 2 (m \in \mathbf{Z}^+)$, then 4 or 5 is the remainder of n and $n + 28$ when divided by 9. So, they cannot be expressed as the sum of the cubes of three positive integers. But

$$n + 2 = (3m-1)^3 + (3m-1)^3 + (3m-1)^3.$$

If $i = 2$, take $n = (3m-1)^3 + 222 (m \in \mathbf{Z}^+)$, then 5 is the remainder of n when divided by 9. So, it cannot be expressed as the sum of the cubes of three positive integers. But

$$n + 2 = (3m-1)^3 + 2^3 + 6^3,$$
$$n + 28 = (3m-1)^3 + 5^3 + 5^3.$$

If $i = 3$, take $n = 216m^3$ $(m \in \mathbf{Z}^+)$. It satisfies the conditions:

$$n = (3m)^3 + (4m)^3 + (5m)^3,$$
$$n + 2 = (6m)^3 + 1^3 + 1^3,$$
$$n + 28 = (6m)^3 + 1^3 + 3^3.$$

This completes the proof.

4 Eight persons join a party.

(1) If there exist three persons who know each other in any group of five, prove that we can find that four persons know each other.

(2) If there exist three persons in a group of six who know each other in a cyclical manner, can we find four persons who know each other in a cyclical manner?

Solution (1) By means of graph theory, use 8 vertices to denote 8 persons. If two persons know each other, we connect them with an edge. With the given condition, there will be a triangle in every induced subgragh with five vertices, while every triangle in the graph belonging to different $\binom{8-3}{2}=\binom{5}{2}=10$ induced subgraghs with five vertices. We know that there are $3\times\binom{8}{5}=3\times 56=168$ edges in total in these triangles, while every edge is computed ten times repeatedly. Thus every vertex is incident with at least $\frac{2\times 168}{8\times 10}>4$ edges. So there exists one vertex A that are incident with at least five edges.

Suppose the vertex A is adjacent with five vetices B, C, D, E, F. By the condition, there exists one triangle in five vertices. Without loss of generality, let $\triangle BCD$ denote the triangle. So there exists one edge between any two vertices in the four vertices A, B, C, D. So the corresponding four persons of the four vertices know each other.

(2) If there exist three persons (in a group of six) who know one another in a cyclical manner, there may not exist four persons who know each other.

For example, let 8 vertices denote 8 persons. If two persons know each other, we join them with an edge. Consider the regular octagon, we link up the 8 shortest diagonals, as desired.

Second Day

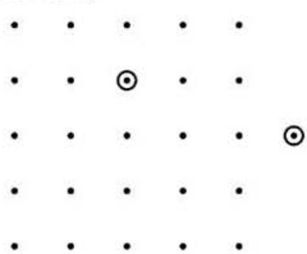
0900 – 1330 August 9, 2006

5 Let $S = \{(a, b) \mid 1 \leqslant a, b \leqslant 5, a, b \in \mathbf{Z}\}$. Let T be the set of integer points in the plane such that for any point P in S, there exists a different point Q in T such that PQ does not contain integer points except P and Q. Find the minimum value of $|T|$, where $|T|$ denotes the number of elements of the finite set T.

Solution We first prove that $|T| \neq 1$.

If $|T| = 1$, let $T = \{Q(x_0, y_0)\}$. We may take point $P(x_1, y_1)$ in S satisfying the conditions: (1) $(x_1, y_1) \neq (x_0, y_0)$, (2) x_1 and x_0 have the same parity, y_1 and y_0 have the same parity. Then, the midpoint of PQ is an integer, which is a contradiction.

If $|T| = 2$, see the following figure satisfying the conditions of the problem:



as desired.

6 Let $M = \{1, 2, \cdots, 19\}$ and $A = \{a_1, a_2, \cdots, a_k\} \subseteq M$. Find the minimum value of k so that there exist a_i, $a_j \in A$ such that if $b \in M$, then $a_i = b$ or $a_i \pm a_j = b$.

Solution By the definition of A, we have $k(k+1) \geqslant 19$, implying $k \geqslant 4$.

If $k = 4$, we have $k(k+1) = 20$. We may assume that $a_1 < a_2 < a_3 < a_4$. Then, $a_4 \geqslant 10$.

(1) If $a_4 = 10$, then $a_3 = 9$, and we have $a_2 = 8$ or 7. If $a_2 = 8$, then 20, $10 - 9 = 1$, $9 - 8 = 1$, impossible. If $a_2 = 7$, then $a_1 = 6$ or 5. Since 20, $10 - 9 = 1$, $7 - 6 = 1$ or 20, $9 - 7 = 2$, $7 - 5 = 2$, impossible.

(2) If $a_4 = 11$, then $a_3 = 8$, and we have $a_2 = 7$ and $a_1 = 6$, impossible.

(3) If $a_4 = 12$, then $a_3 = 7$, and we have $a_2 = 6$ and $a_1 = 5$, impossible.

(4) If $a_4 = 13$, then $a_3 = 6$, $a_2 = 5$, $a_1 = 4$, impossible.

(5) If $a_4 = 14$, then $a_3 = 5$, $a_2 = 4$, impossible.

(6) If $a_4 = 15$, then $a_3 = 4$, $a_2 = 3$, $a_1 = 2$, impossible.

(7) If $a_4 = 16$, then $a_3 = 3$, $a_2 = 2$, $a_1 = 1$, impossible.

(8) If $a_4 \geqslant 17$, impossible.

So $k \geqslant 5$. Let $A = \{1, 3, 5, 9, 16\}$, then A satisfies the conditions of the problem. Therefore $k_{\min} = 5$.

7 Let $x_i > 0$, and $k \geqslant 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \sum_{i=1}^{n} x_i \leqslant \sum_{i=1}^{n} \frac{x_i^{k+1}}{1+x_i} \sum_{i=1}^{n} \frac{1}{x_i^k}.$$

Proof I Observe that the above inequality is equivalent to

$$\sum_{i=1}^{n} \frac{x_i^{k+1}}{1+x_i} \sum_{i=1}^{n} \frac{1}{x_i^k} - \sum_{i=1}^{n} \frac{1}{1+x_i} \sum_{i=1}^{n} x_i \geqslant 0. \qquad ①$$

The left-hand side of ① is equal to

$$\begin{aligned}
&\sum_{i \neq j} \frac{x_i^{k+1}}{1+x_i} \cdot \frac{1}{x_j^k} - \sum_{i \neq j} \frac{x_j}{1+x_i} \\
&= \sum_{i \neq j} \frac{x_i^{k+1} - x_j^{k+1}}{(1+x_i)x_j^k} \\
&= \frac{1}{2} \sum_{i \neq j} \left[\frac{x_i^{k+1} - x_j^{k+1}}{(1+x_i)x_j^k} + \frac{x_j^{k+1} - x_i^{k+1}}{(1+x_j)x_i^k} \right] \\
&= \frac{1}{2} \sum_{i \neq j} (x_i^{k+1} - x_j^{k+1}) \frac{(1+x_j)x_i^k - (1+x_i)x_j^k}{(1+x_j)(1+x_i)x_i^k x_j^k} \\
&= \frac{1}{2} \sum_{i \neq j} (x_i^{k+1} - x_j^{k+1}) \frac{(x_i^k - x_j^k) + x_i x_j (x_i^{k-1} - x_j^{k-1})}{(1+x_j)(1+x_i)x_i^k x_j^k} \\
&\geqslant 0.
\end{aligned}$$

Proof Ⅱ We may assume that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n > 0$. Then, we have

$$\frac{1}{x_1^k} \leqslant \frac{1}{x_2^k} \leqslant \cdots \leqslant \frac{1}{x_n^k}, \qquad ①$$

$$\frac{x_1^k}{1+x_1} \geqslant \frac{x_2^k}{1+x_2} \geqslant \cdots \geqslant \frac{x_n^k}{1+x_n}. \qquad ②$$

By the *Chebyschev* Inequality, the left-hand side of original inequality is equal to

$$\begin{aligned}
&\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right)(x_1 + x_2 + \cdots + x_n) \\
&= \left(\frac{1}{x_1^k} \cdot \frac{x_1^k}{1+x_1} + \frac{1}{x_2^k} \cdot \frac{x_2^k}{1+x_2} + \cdots + \frac{1}{x_n^k} \cdot \frac{x_n^k}{1+x_n}\right) \\
&\quad \times (x_1 + x_2 + \cdots + x_n) \\
&\leqslant \left(\frac{1}{x_1^k} + \frac{1}{x_2^k} + \cdots + \frac{1}{x_n^k}\right)\left(\frac{x_1^k}{1+x_1} + \frac{x_2^k}{1+x_2} + \cdots + \frac{x_n^k}{1+x_n}\right)
\end{aligned}$$

$$\begin{aligned}
&\times (x_1 + x_2 + \cdots + x_n)/n \\
&\leqslant \left(x_1 \cdot \frac{x_1^k}{1+x_1} + x_2 \cdot \frac{x_2^k}{1+x_2} + \cdots + x_n \cdot \frac{x_n^k}{1+x_n}\right) \\
&\quad \times \left(\frac{1}{x_1^k} + \frac{1}{x_2^k} + \cdots + \frac{1}{x_n^k}\right) \\
&= \left(\frac{x_1^{k+1}}{1+x_1} + \frac{x_2^{k+1}}{1+x_2} + \cdots + \frac{x_n^{k+1}}{1+x_n}\right)\left(\frac{1}{x_1^k} + \frac{1}{x_2^k} + \cdots + \frac{1}{x_n^k}\right) \\
&= \sum_{i=1}^{n} \frac{x_i^{k+1}}{1+x_i} \sum_{i=1}^{n} \frac{1}{x_i^k}.
\end{aligned}$$

8 Let p be a prime number greater than 3. Prove that there exist integers $a_1, a_2, \cdots, a_t$ that satisfy the following conditions:

(a)

$$-\frac{p}{2} < a_1 < a_2 < \cdots < a_t \leqslant \frac{p}{2},$$

(b)

$$\frac{p-a_1}{|a_1|} \cdot \frac{p-a_2}{|a_2|} \cdot \cdots \cdot \frac{p-a_t}{|a_t|} = 3^m,$$

where m is a positive integer.

Proof By the Division Algorithm, there exists unique integers q and r such that $p = 3q + r$, where $0 < r < 3$.

Taking $b_0 = r$, then

$$\frac{p-b_0}{|b_0|} = \frac{3^{c_0} \cdot b_1^*}{|b_0|}, \text{ where } 3 \nmid b_1^* \text{ and } 0 < b_1^* < \frac{p}{2}.$$

Taking $b_1 = \pm b_1^*$ such that $b_1 \equiv p \pmod 3$, then

$$\frac{p-b_1}{|b_1|} = \frac{3^{c_1} \cdot b_2^*}{b_1^*}, \text{ where } 3 \nmid b_2^* \text{ and } 0 < b_2^* < \frac{p}{2}.$$

Taking $b_2 = \pm b_2^*$ such that $b_2 \equiv p \pmod 3$, then

$$\frac{p - b_2}{|b_2|} = \frac{3^{c_2} \cdot b_3^*}{b_2^*}, \text{ where } 3 \nmid b_3^* \text{ and } 0 < b_3^* < \frac{p}{2}.$$

Repeating this process, we get

$$b_0, b_1, \cdots, b_p.$$

Since these $p+1$ integers are in the interval $\left(-\frac{p}{2}, \frac{p}{2}\right)$, a certain integer occurs twice. Suppose $b_i = b_j$, $i < j$, and b_i, b_{i+1}, $\cdots$, b_{j-1} are distinct. So,

$$\frac{p - b_i}{|b_i|} \cdot \frac{p - b_{i+1}}{|b_{i+1}|} \cdot \cdots \cdot \frac{p - b_{j-1}}{|b_{j-1}|}$$
$$= \frac{3^{c_i} \cdot b_{i+1}^*}{b_i^*} \cdot \frac{3^{c_{i+1}} \cdot b_{i+2}^*}{b_{i+1}^*} \cdot \cdots \cdot \frac{3^{c_{j-1}} \cdot b_j^*}{b_{j-1}^*}.$$

Since $b_i = b_j$, then $b_i^* = b_j^*$. So the above expression equals to

$$3^{c_i + c_{i+1} + \cdots + c_{j-1}} = 3^n, \ n > 0.$$

Put b_i, b_{i+1}, $\cdots$, b_{j-1} in ascending order, as desired.

2007 (Wuhan, Hubei)

August 13 and 14, 2007

1 A positive integer m is called good, if there is a positive integer n such that m is the quotient of n over the number of positive integer divisors of n (including 1 and n itself). Prove that 1, 2, $\cdots$, 17 are good numbers and that 18 is not a good number.

Proof For positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself).

Firstly, note that 1 and 2 are good, since $1 = \frac{2}{d(2)}$ and $2 = \frac{8}{d(8)}$.

Secondly, we note that if p is an odd prime, then p is good. This is because $d(8p) = 8$. In particular, 3, 5, 7, 11, 13, 17 are good numbers.

Thirdly, we note that p is an odd prime, then $2p$ is good. This is because $d(2^2 \cdot 3^2 p) = 3 \cdot 3 \cdot 2$. In particular, 6, 10, 14 are good numbers.

Fourthly, we note that

$$4 = \frac{36}{d(36)},\ 8 = \frac{96}{d(96)},\ 9 = \frac{108}{d(108)},$$
$$12 = \frac{240}{d(240)},\ 15 = \frac{360}{d(360)},\ 16 = \frac{128}{d(128)}.$$

Thus, the numbers 1, 2, $\cdots$, 17 are good.

Finally, we prove that 18 is not good. We approach indirectly by assuming that $18 = \frac{n}{d(n)}$ or $n = 18d(n)$ for $n = 2^a \cdot 3^{b+1} \cdot p_1^{k_1} \cdots p_m^{k_m}$ (where $p_1 < \cdots < p_m$ are prime numbers greater than 3 and a, b, k_1, $\cdots$, k_m are positive integers); that is,

$$2^{a-1} \cdot 3^{b-1} \cdot p_1^{k_1} \cdots p_m^{k_m} = (a+1)(b+2)(k_1+1)\cdots(k_m+1). \qquad ①$$

For every odd prime p and every positive integer k, we can show (by an easy induction on k) that

$$p^k > k+1. \qquad ②$$

Combining the last two relations, we deduce that

$$2^{a-1} \cdot 3^{b-1} < (a+1)(b+2)$$

or

$$f(a) = \frac{2^{a-1}}{a+1} < \frac{b+2}{3^{b-1}} = g(b).$$

It is easy to prove that $f(1) = \frac{1}{2}$, $f(2) = \frac{2}{3}$, $f(3) = 1$, $f(4) = \frac{8}{5}$, $f(5) = \frac{16}{6}$, and $f(a) \geqslant \frac{32}{7} > 4$ for $a \geqslant 6$. It is also easy to prove that $g(1) = 3$, $g(2) = \frac{4}{3}$, $g(3) < \frac{5}{9}$, and $g(b) < \frac{2}{9}$ for $b \geqslant 4$. Thus ① holds only if $b \leqslant 3$.

If $b = 3$, then $(a, b) = (1, 3)$, and ① becomes

$$9p_1^{k_1} \cdots p_m^{k_m} = 10(k_1 + 1)(k_m + 1),$$

implying that $p_1 = 5$. Since ② and $\frac{p_1^{k_1}}{k_1 + 1} = \frac{5^{k_1}}{k_1 + 1} \geqslant \frac{5}{2}$ for positive integer $k_1 \geqslant 1$, we can easily see that there is no solution in this case.

If $b = 2$, then $(a, b) = (1, 2)$, $(2, 2)$, $(3, 2)$, and ① becomes

$$3 \cdot 2^{a-1} p_1^{k_1} \cdots p_m^{k_m} = 4(a+1)(k_1+1)\cdots(k_m+1).$$

We deduce that 4 divides 2^{a-1} or $a \geqslant 3$. Hence a must be equal to 3. But then $4(a+1) = 16$ divides 2^{a-1}, which is impossible.

If $b = 1$, then $(a, b) = (1, 1)$, $(2, 1)$, $(3, 1)$, $(4, 1)$, $(5, 1)$, and ① becomes

$$2^{a-1} p_1^{k_1} \cdots p_m^{k_m} = 3(a+1)(k_1+1)\cdots(k_m+1),$$

which is impossible since p_i are primes greater than 3.

In all the cases, we cannot find n satisfying the condition $18=\frac{n}{d(n)}$; that is, 18 is not a good number.

Note For primes $p \geqslant 5$, both $3p$ and p^2 are good, since $d(2^2 \cdot 3p)=3 \cdot 2 \cdot 2$ and $d(2^3 \cdot 3 \cdot p^2)=4 \cdot 2 \cdot 3$.

2 Let ABC be an acute triangle. Points D, E and F lie on segments BC, CA and AB respectively, and each of the three segments AD, BE and CF contains the circumcenter of ABC. Prove that if any two of the ratios

$$\frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB}, \frac{BF}{FA}, \frac{AE}{EC}, \frac{CD}{DB}$$

are integers, then triangle ABC is isosceles.

Proof I Note that there are $\binom{6}{2}=15$ possible pairs of ratios among the six given in the problem statement. These pairs are of two types: (i) Three of these pairs are reciprocal pairs involving segments from just one side of triangle ABC. (ii) The other 12 pairs involve segments from two sides of the triangle. We first consider the former case.

(a) If $\frac{CD}{DB}$ and $\frac{BD}{DC}$ are both integers, then both of these ratios must be 1 and $BD=DC$. Then in triangle ABC, AD is the median from A and D, because AD contains the circumcenter, is also the perpendicular bisector of segment BC. It then follows that $AB=AC$ and the triangle is isosceles. Similarly, if $\frac{CE}{EA}$ and $\frac{AE}{EC}$ are both integers or $\frac{AF}{FB}$ and $\frac{BF}{FA}$ are both integers, then triangle ABC is isosceles.

(b) Let O be the circumcenter of triangle ABC, and let

$\angle CAB = \alpha$, $\angle ABC = \beta$, and $\angle BCA = \gamma$. We show that any of the ratios can be written in the form $\frac{\sin 2x}{\sin 2y}$ where x and y are two of α, β, γ. Since ABC is acute, $0^\circ < \alpha, \beta, \gamma < 90^\circ$ and O lies in the interior. Hence $\angle AOB = 2\gamma$, $\angle BOC = 2\alpha$, and $\angle COA = 2\beta$. Applying the sine rule to triangles BOD and COD gives

$$\frac{BD}{\sin \angle BOD} = \frac{BO}{\sin \angle BDO}$$

and

$$\frac{CD}{\sin \angle COD} = \frac{CO}{\sin \angle CDO}.$$

Next note that $BO = CO$ and that

$$\begin{aligned}\angle BDO + \angle CDO &= 180^\circ \\ &= \angle BOD + \angle AOB \\ &= \angle COD + \angle AOC.\end{aligned}$$

It follows that

$$\frac{BD}{\sin 2\gamma} = \frac{BD}{\sin \angle BOD} = \frac{CD}{\sin \angle COD} = \frac{CD}{\sin 2\beta},$$

giving $\frac{BD}{CD} = \frac{\sin 2\gamma}{\sin 2\beta}$. Similarly, $\frac{CE}{EA} = \frac{\sin 2\alpha}{\sin 2\gamma}$ and $\frac{AF}{FB} = \frac{\sin 2\beta}{\sin 2\alpha}$.

Now assume that one of the twelve type (ii) pairs of ratios consists of two integers. Then there are positive integers m and n (with $m \leqslant n$) such that

$$\sin 2x = m \sin 2z \quad \text{and} \quad \sin 2y = n \sin 2z$$

or

$$\sin 2z = m \sin 2x \quad \text{and} \quad \sin 2z = n \sin 2y$$

for some choice of x, y, z with $\{x, y, z\} = \{\alpha, \beta, \gamma\}$. Without loss of generality we may assume that

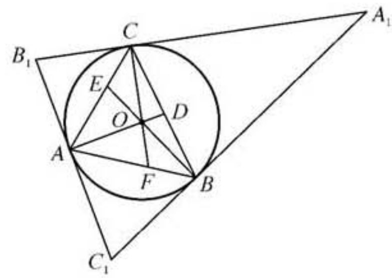
$$\sin 2\alpha = m \sin 2\gamma \quad \text{and} \quad \sin 2\beta = n \sin 2\gamma$$

or

$$\sin 2\gamma = m \sin 2\alpha \quad \text{and} \quad \sin 2\gamma = n \sin 2\beta \qquad ①$$

for some positive integers m and n.

Note that there is a triangle with angles $180^\circ - 2\alpha$, $180^\circ - 2\beta$, and $180^\circ - 2\gamma$. (It is easy to check that each of these angles is in the interval $(0^\circ, 180^\circ)$ and that they sum to 180°.) Furthermore, a triangle



with these angles can be constructed by drawing the tangents to the circumcircle of ABC at each of A, B and C. Denote this triangle by $A_1B_1C_1$ where A_1 is the intersection of the tangents at B and C, B_1 is the intersection of the tangents at C and A, and C_1 is the intersection of the tangents at A and B. Applying the sine rule to triangle $A_1B_1C_1$ and by ① we find

$$\begin{aligned} A_1B_1 : B_1C_1 : C_1A_1 &= \sin \angle C_1 : \sin \angle A_1 : \sin \angle B_1 \\ &= \sin 2\gamma : \sin 2\alpha : \sin 2\beta, \end{aligned}$$

that is,

$$A_1B_1 : B_1C_1 : C_1A_1 = 1 : m : n$$

or

$$A_1B_1 : B_1C_1 : C_1A_1 = mn : n : m. \qquad ②$$

By the triangle inequality, if follows that $1 + m < n$ (that is,

$m = n$) or $n + m > nm$ (that is, $(n-1)(m-1) < 1$ and $m = 1$). We deduce that either $\sin 2\alpha = \sin 2\beta$ or $\sin 2\gamma = \sin 2\alpha$. But then either $\frac{AF}{FB} = \frac{BF}{FA}$ or $\frac{CE}{EA} = \frac{AE}{EC}$, by case (a), triangle ABC is isosceles.

Proof Ⅱ (We maintain the notations used in Proof Ⅰ.) We only consider those 12 pairs of ratios of type (ii). Without loss of generality we may assume that each of the following sets $\left\{\frac{BD}{DC}, \frac{CD}{DB}\right\}$ and $\left\{\frac{AF}{FB}, \frac{BF}{FA}\right\}$ has an element taking integer values. By symmetry, we consider three cases:

(a) In this case, we assume that $\frac{BD}{DC} = m$ and $\frac{BF}{FA} = n$ for some positive integers m and n. Applying *Menelaus's* theorem to line AOD and triangle BCF yields

$$\frac{AO}{OD} \cdot \frac{DC}{CB} \cdot \frac{BF}{FA} = 1$$

or

$$\frac{AO}{OD} = \frac{CB}{DC} \cdot \frac{FA}{BF} = \frac{m+1}{n}.$$

Likewise, applying *Menelaus's* theorem to line COF and triangle BAD yields $\frac{CO}{OF} = \frac{n+1}{m}$.

Since triangle ABC is acute, O lies on segment AD and CF with $AO > OD$ and $CO > OF$. Hence $m + 1 > n$ and $n + 1 > m$, implying that $m - 1 < n < m + 1$. Since m and n are integers, we must have $m = n$. It is then not difficult to see that A and C are symmetric with respect to line BF and triangle ABC is isosceles with $AB = CB$.

(b) In this case, we assume that $\frac{CD}{DB} = m$ and $\frac{AF}{FB} = n$ for some positive integers m and n. Applying *Ceva'* theorem gives

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \quad \text{or} \quad \frac{CE}{EA} = \frac{m}{n}.$$

Applying *Menelaus's* theorem to line BOE and triangle ACF yields

$$\frac{BO}{OE} \cdot \frac{EC}{CA} \cdot \frac{AF}{FB} = 1$$

or

$$\frac{BO}{OE} = \frac{FB}{AF} \cdot \frac{CA}{EC} = \frac{m+n}{mn}.$$

Since triangle ABC is acute, O lies on segment BE and $EO < BO$. Hence $m + n \geqslant mn$ or $(m-1)(n-1) \leqslant 1$. Since m and n are positive integers, we deduce that one of m and n is equal to 1; that is, either $AF = FB$ or $BD = DC$. In either case, triangle ABC is isosceles.

(c) In this case, we assume that $\frac{CD}{DB} = m$ and $\frac{BF}{FA} = n$ for some positive integers m and n. Applying *Ceva's* theorem gives

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \quad \text{or} \quad \frac{CE}{EA} = mn.$$

We can treat this case as either case (a) (by viewing C in place of B) or case (b) (by viewing A in place of B).

3 Let n be an integer greater than 3, and let $a_1, a_2, \cdots, a_n$ be nonnegative real numbers with $a_1 + a_2 + \cdots + a_n = 2$.

Determine the minimum value of

$$\frac{a_1}{a_2^2+1}+\frac{a_2}{a_3^2+1}+\cdots+\frac{a_n}{a_1^2+1}.$$

Solution The answer is $\frac{3}{2}$.

The given problem is equivalent to finding the minimum value of

$$\begin{aligned} m &= 2-\left(\frac{a_1}{a_2^2+1}+\frac{a_2}{a_3^2+1}+\cdots+\frac{a_n}{a_1^2+1}\right) \\ &= \left(a_1-\frac{a_1}{a_2^2+1}\right)+\left(a_2-\frac{a_2}{a_3^2+1}\right)+\cdots+\left(a_n-\frac{a_n}{a_1^2+1}\right) \\ &= \frac{a_1a_2^2}{a_2^2+1}+\frac{a_2a_3^2}{a_3^2+1}+\cdots+\frac{a_na_1^2}{a_1^2+1}. \end{aligned}$$

Since $a_i^2+1 \geqslant 2a_i$, we have

$$m \leqslant \frac{a_1a_2+a_2a_3+\cdots+a_na_1}{2}.$$

Our result follows from the following well-known fact:

$$\begin{aligned} & f(a_1, \cdots, a_n) \\ = & (a_1+\cdots+a_n)^2-4(a_1a_2+a_2a_3+\cdots+a_na_1) \\ \geqslant & 0 \end{aligned} \qquad ①$$

for integers $n \geqslant 4$ and nonnegative real numbers $a_1, a_2, \cdots, a_n$.

To prove this fact, we use induction on n. For $n=4$, ① becomes

$$\begin{aligned} & f(a_1, a_2, a_3, a_4) \\ = & (a_1+a_2+a_3+a_4)^2-4(a_1a_2+a_2a_3+a_3a_4+a_4a_1) \\ = & (a_1+a_2+a_3+a_4)^2-4(a_1+a_3)(a_2+a_4), \end{aligned}$$

which is nonnegative by the AM-GM inequality.

Assume that ① is true for $n=k$ for some integer $k \geqslant 4$.

Consider the case when $n = k+1$. By (cyclic) symmetry in ①, we may assume that $a_{k+1} = \min\{a_1, a_2, \cdots, a_{k+1}\}$. By the induction hypothesis, it suffices to show that

$$D = f(a_1, \cdots, a_k, a_{k+1}) - f(a_1, \cdots, a_{k-1}, a_k + a_{k+1}) \geqslant 0.$$

Note that

$$\begin{aligned}\frac{D}{4} &= -(a_1a_2 + \cdots + a_ka_{k+1} + a_{k+1}a_1) + [a_1a_2 + \cdots \\ &\quad + a_{k-1}(a_k + a_{k+1}) + (a_k + a_{k+1})a_1] \\ &= a_{k-1}a_{k+1} + a_ka_1 - a_ka_{k+1} \\ &= a_{k-1}a_{k+1} + (a_1 - a_{k+1})a_k \\ &\geqslant 0,\end{aligned}$$

completing our proof.

From the above result, we can get

$$\begin{aligned}&\frac{1}{2}(a_1a_2 + a_2a_3 + \cdots + a_na_1) \\ \leqslant &\frac{1}{8}(a_1 + a_2 + \cdots + a_n)^2 \\ = &\frac{1}{8} \times 2^2 \\ = &\frac{1}{2},\end{aligned}$$

therefore

$$\frac{a_1a_2^2}{a_2^2+1} + \frac{a_2a_3^2}{a_3^2+1} + \cdots + \frac{a_na_1^2}{a_1^2+1} \leqslant \frac{1}{2},$$

that is $m \geqslant \frac{3}{2}$. When $a_1 = a_2 = 1$ and $a_3 = \cdots = a_n = 0$, $m = \frac{3}{2}$ hold.

So the minimum value is $\frac{3}{2}$.

4. The set S consists of $n > 2$ points in the plane. The set P consists of m lines in the plane such that every line in P is an axis of symmetry of S. Prove that $m \leqslant n$, and determine when equality holds.

Proof (a) Denote the n points as $A_1, A_2, \cdots, A_n$, and $A_i = (x_i, y_i)$, $(i = 1, 2, \cdots, n)$ in the coordinate system. It is obvious that the equality $\sum_{i=1}^{n} \overrightarrow{BA_i} = \vec{0}$ holds if and only if $B = \left(\frac{1}{n}\sum_{i=1}^{n} x_i, \frac{1}{n}\sum_{i=1}^{n} y_i\right)$, which means there exists only one point B in the plane such that $\sum_{i=1}^{n} \overrightarrow{BA_i} = \vec{0}$, and we call it "center of set S".

If we take any line p in the set P as the x axis of the coordinate system, then $\sum_{i=1}^{n} y_i = 0$, that is, B belongs to p. Thus, every line in P passes through point B.

(b) Let

$$F = \{(x, y, p) \mid x, y \in S, p \in P, \\ p \text{ is symmetric axis of } x, y\},$$

$$F_1 = \{(x, y, p) \in F \mid x \neq y\},$$

$$F_2 = \{(x, y, p) \in F \mid x \text{ belongs to } p\}.$$

Then,

$$F = F_1 \cup F_2, F_1 \cap F_2 = \varnothing. \tag{1}$$

Considering any line p in the set P, and any point x in the set S, we could get that x has only one symmetric point y, with p as the symmetry axis. Therefore, there are n corresponding arrays (x, y, p) for any p, and

$$|F| = mn. \quad ②$$

As to the array (x, y, p) in F_1, since there is only one symmetric axis for the different points x and y, then

$$|F_1| \leqslant \{(x, y) \mid x, y \in S, x \neq y\} = 2\binom{n}{2} = n(n-1). \quad ③$$

As to the array in F_2.

When any point in S belongs to no more than one line p, then

$$|F_2| \leqslant \{x \mid x \in S\} = n. \quad ④$$

From ①-④, we get

$$mn \leqslant n(n-1) + n,$$

that is, $m \leqslant n$.

When there exists one point that belongs to two lines of P, then as proved in (a), it must be the center of set B. Considering the set $S' = S\backslash\{B\}$, since every line in P is still the symmetric axis of the points in S', then we get

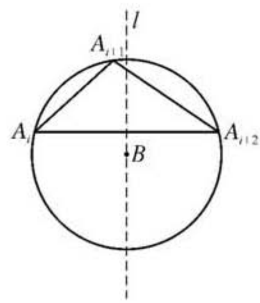
$$m \leqslant |S'| = n - 1.$$

Thus, we obtain $m \leqslant n$.

(c) When $m = n$, the equalities ③, ④ in (b) hold simultaneously. So, perpendicular bisector of a line segment joining any two points in S belongs to P, and any point in S belongs to one line in P, while the 'center of set' B is not in S.

Now, we could first prove all the $BA_i (i = 1, 2, \cdots, n)$ are equal. Otherwise, if there exist j, k $(1 \leqslant j < k \leqslant n)$ such that $BA_j \neq BA_k$, then the symmetric axis of A_jA_k does not pass through B, which is contradictory to (a). Thus, A_1, A_2, $\cdots$, A_n are all on the circle with B as its center, that is, $\odot B$. We could suppose A_1, A_2, $\cdots$, A_n are arranged clockwise for convenience.

Then, we could also prove that A_1, A_2, $\cdots$, A_n are n points dividing $\odot B$ into equal parts. Otherwise, if there exists i ($i = 1, 2, \cdots, n$) such that $A_iA_{i+1} \neq A_{i+1}A_{i+2}$ (let $A_{n+1} = A_1$, $A_{n+2} = A_2$), we could suppose $A_iA_{i+1} < A_{i+1}A_{i+2}$. Then as shown in the figure, the symmetric axis $l \in P$, but the symmetric point of A_{i+1} is on the arc $\overparen{A_{i+1}A_{i+2}}$ (excluding the point A_{i+1}, A_{i+2}). However, this is in contradict with the condition that A_{i+1} and A_{i+2} are adjacent.



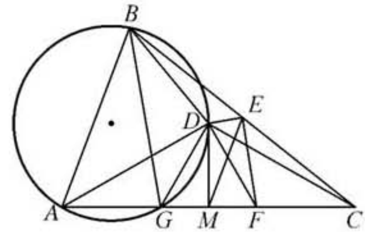
Hence, the points in S are the vertices of a regular n-gon (n-sided polygon), when $m = n$. On the other hand, it is obvious that the regular n-gon has exactly n symmetric axes. Therefore, the points in S are the vertices and the lines in P are the symmetric axes of the regular n-gon if and only if $m = n$.

5 Point D lies inside triangle ABC such that $\angle DAC = \angle DCA = 30°$ and $\angle DBA = 60°$. Point E is the midpoint of segment BC. Point F lies on segment AC with $AF = 2FC$. Prove that $DE \perp EF$.

Proof Ⅰ Let G and M be the midpoints of segments AF and AC respectively.

In right triangle ADM, $\angle ADM = 60°$ and $AM = \sqrt{3}DM$.

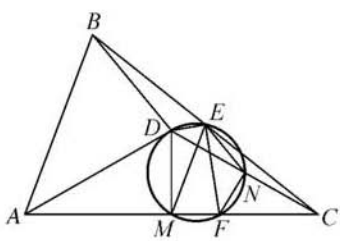
Note that $AM = 3GM$. Hence $DM = \sqrt{3}GM$ and $DG = 2GM = AG = GF$. By symmetry, $DF = GF$ and DFG is an isosceles triangle.



In triangle ADG, $AG = DG$ and $\angle ADG = 30°$. Since $\angle ABD = \angle DGF = 60°$, $ABDG$ is concyclic, implying that $\angle ABG = \angle ADG = 30°$. Note that EF and EM are midlines in triangles BGC and BAC respectively. In particular, $EF \parallel BG$ and $EM \parallel BA$, implying that $\angle MEF = \angle ABG = 30°$.

Therefore, $\angle MEF = \angle MDF = 30°$ and $MDEF$ is concyclic, from which it follows that $\angle DEF = 180° - \angle DMF = 90°$.

Proof Ⅱ (We maintain the notations of the first proof.) Let N be the midpoint of segment CD. Then EN and ME are the respective midlines in triangles BDC and ABC. In particular, $EN \parallel BD$ and $EM \parallel BA$, implying that $\angle MEN = \angle ABD = 60°$. Thus, $\angle MDN = \angle MEN = 60°$, that is, $MDEN$ is concyclic.



It is easy to compute that $AC = \sqrt{3}CD$, $CF = \sqrt{3}CN$ and $\frac{CD}{CM} = \frac{CF}{CN}$. Thus, triangles CNF and CMD are similar. Consequently, we deduce that $\angle CNF = \angle FMD = 90°$, that is, $MDNF$ is concyclic.

Therefore, $MDENF$ is concyclic (with DF as its diameter) and $\angle DEF = 90°$.

6 For nonnegative real numbers a, b, c with $a + b + c = 1$, prove that

$$\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c}\leqslant\sqrt{3}.$$

Proof Ⅰ Without loss of generality, we may assume that $b \geqslant c$. We set $\sqrt{b} = x+y$ and $\sqrt{c} = x-y$ for some nonnegative real numbers x and y. Hence $b-c=4xy$ and $a=1-2x^2-2y^2$. It follows that

$$\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c} = \sqrt{1-2x^2-2y^2+4x^2y^2}+2x. \qquad ①$$

Note that $2x = \sqrt{b}+\sqrt{c}$, implying that

$$4x^2 = (\sqrt{b}+\sqrt{c})^2 \leqslant 2b+2c \leqslant 2,$$

by the AM - GM inequality. Thus, $4x^2y^2 \leqslant 2y^2$ and

$$1-2x^2-2y^2+4x^2y^2 \leqslant 1-2x^2.$$

Substituting the last inequality into ① yields

$$\begin{aligned}\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c} &\leqslant \sqrt{1-2x^2}+2x\\ &= \sqrt{1-2x^2}+x+x\\ &\leqslant\sqrt{3},\end{aligned}$$

by the *Cauchy-Schwarz* inequality.

Proof Ⅱ Let $a=u^2$, $b=v^2$, and $c=w^2$. Then $u^2+v^2+w^2=1$ and the desired inequality becomes

$$\sqrt{u^2+\frac{(v^2-w^2)^2}{4}}+v+w\leqslant\sqrt{3}. \qquad ②$$

Note that

$$\begin{aligned}
u^2+\frac{(v^2-w^2)^2}{4} &= 1-(v^2+w^2)+\frac{(v^2-w^2)^2}{4}\\
&= \frac{4-4(v^2+w^2)+(v^2-w^2)^2}{4}\\
&= \frac{4-4(v^2+w^2)+(v^2+w^2)^2-4v^2w^2}{4}\\
&= \frac{(2-v^2-w^2)^2-4v^2w^2}{4}\\
&= \frac{(2-v^2-w^2-2vw)(2-v^2-w^2+2vw)}{4}\\
&= \frac{[2-(v+w)^2][2-(v-w)^2]}{4}\\
&\leqslant 1-\frac{(v+w)^2}{2}.
\end{aligned}$$

(Note that $(v+w)^2 \leqslant 2(v^2+w^2) \leqslant 2$) Substitute the above equation into ② and it gives

$$\sqrt{1-\frac{(v+w)^2}{2}}+v+w \leqslant \sqrt{3}.$$

Set $\frac{v+w}{2}=x$. We can rewrite the above inequality as

$$\sqrt{1-2x^2}+2x \leqslant \sqrt{3},$$

and we can complete the proof as we did in the first proof.

Note The second proof reveals the motivation of the substitution used in the first proof.

7 Let a, b, c be integers each with absolute value less than or equal to 10. The cubic polynomial

$$f(x)=x^3+ax^2+bx+c$$

satisfies the property

$$|f(2+\sqrt{3})|<0.0001.$$

Determine if $2+\sqrt{3}$ is a root of f.

Solution We attempt to arrive at a contradiction by assuming that $2+\sqrt{3}$ is not a root of f. We need to evaluate

$$\begin{aligned} f(2+\sqrt{3}) &= (2+\sqrt{3})^3 + a(2+\sqrt{3})^2 + b(2+\sqrt{3}) + c \\ &= (26+7a+2b+c) + (15+4a+b)\sqrt{3}. \end{aligned}$$

Let $m = 26+7a+2b+c$ and $n = 15+4a+b$. Then $|m| < 130$ and $n \leqslant 65$. It follows that

$$|m - n\sqrt{3}| \leqslant 130 + 65\sqrt{3} < 260.$$

Thus,

$$|f(2+\sqrt{3})| = |m + n\sqrt{3}| = \left|\frac{m^2 - 3n^2}{m - n\sqrt{3}}\right|.$$

By our assumption, $f(2+\sqrt{3})$ is nonzero. Hence $m + n\sqrt{3} \neq 0$. Since m and n are integers, and $\sqrt{3}$ is irrational, $|m^2 - 3n^2| \geqslant 1$. It follows that

$$|f(2+\sqrt{3})| = \left|\frac{m^2 - 3n^2}{m - n\sqrt{3}}\right| \geqslant \frac{1}{260} \geqslant 0.001,$$

which is a contradiction. Hence our assumption was wrong and $2+\sqrt{3}$ is a root of f.

8 In a round robin chess tournament each player plays with every other player exactly once. The winner of each game gets 1 point and the loser gets 0 point. If the game ends in a tie, each player gets 0.5 point. Given a positive integer m, a tournament is said to have property $P(m)$ if the following holds: among every set S of m players, there is one player who won all his games against the other $m-1$

players in S and one player who lost all his games against the other $m-1$ players in S.

For a given integer $m \geqslant 4$, determine the minimum value of n (as a function of m) such that the following holds: in every n-player round robin chess tournament with property $P(m)$, the final scores of the n players are all distinct.

Solution Note that if there are $2m-4$ players, we can label them

$$a_1, a_2, \cdots, a_{m-3}, A_{m-2}, B_{m-2}, a_{m-1}, \cdots, a_{2m-5},$$

and assume that player P_i beats player P_j if and only if $i > j$, and A_{m-2} and B_{m-2} are in a tie. It is easy to see that in the group of m players, there exists a unique player P_i with the maximum index i ($m-1 \leqslant i \leqslant 2m-5$, and this player won all games against other players in the group), and there exists a unique player P_i with the minimum index j ($1 \leqslant i \leqslant m-3$, and this player lost all games against other players in the group). Hence this tournament has property $P(m)$ and not all players have distinct total points. If $n < 2m-4$, we can then build a similar tournament by taking players away from both ends index-wise). Hence the answer is greater than $2m-3$. It suffices to show the following claim:

If there are $2m-3$ players in a tournament with property $P(m)$, then the players must have distinct total final score.

In a group, if a player won (or lose) all games against the rest of the players in the group, we call this player the winner (or loser) of the group. If a player won (or lose) all his games in the tournament, we call this player the complete winner (or complete loser). We establish the following lemmas.

Lemma 1 *In an n-player ($n \geqslant m$) tournament with property $P(m)$, there is a complete winner.*

Proof: We implement an induction on n. If $n = m$, the statement is trivial. Now assume that the statement is true for some $n = k$ ($k \geqslant m$), we consider a ($k+1$)-player tournament with property $P(m)$. Let a_1, $\cdots$, a_{k+1} denote the players. By the induction hypothesis, we may assume that a_{k+1} is the winner in the group a_2, $\cdots$, a_{k+1}. We consider three cases:

(a) If a_{k+1} won the game against a_1, then a_{k+1} is the complete winner;

(b) If a_{k+1} tied the game against a_1, then the group a_1, a_2, $\cdots$, a_{k-1}, a_{k+1} has no winner, violating the condition that the tournament has property $P(m)$;

(c) If a_{k+1} lose the game against a_1, then the group $\{a_1, a_2, \cdots, a_{k-1}, a_k, a_{k+1}\}\setminus\{a_i\}$ $(2 \leqslant i \leqslant k)$ has a winner, and this winner can only be a_1. Thus a_1 is the complete winner.

Combining the three cases, we find a complete winner in the tournament, hence our induction is complete.

In exactly the same way, we can prove that

Lemma 2 *In an n-player ($n \geqslant m$) tournament with property $P(m)$, there is a complete loser.*

Now we are ready to prove our claim in a similar manner.

China Western Mathematical Olympiad

2006 (Yingtan, Jiangxi)

The 6th China Western Mathematical Olympiad was held on 5—10 November, 2006 in Yingtan, Jiangxi, China. The event was hosted by Jiangxi Mathematical Society and Yingtan No. 1 middle school.

The competition committee comprised: Leng Gangsong, Wu Jiangpin, Xiong Bin, Li Weigu, Li Shenghong, Wang Jianwei, Zhu Huawei, Tao Pingsheng, Liu Shixiong, Bian Hongping, Feng Zhigang.

First Day

0800 - 1200 November 4, 2006

1 Let n ($\geqslant 2$) be a positive integer and $a_1, a_2, \cdots, a_n \in (0, 1)$. Find the maximum value of the sum

$$\sum_{i=1}^{n} \sqrt[6]{a_i(1-a_{i+1})}$$

where $a_{n+1} = a_1$.

Solution By the AM - GM Inequality, we deduce that

$$\begin{aligned} & \sqrt[6]{a_i(1-a_{i+1})} \\ = & 2^{\frac{4}{6}} \sqrt[6]{a_i(1-a_{i+1}) \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} \\ \leqslant & 2^{\frac{2}{3}} \cdot \frac{1}{6} \cdot (a_i + 1 - a_{i+1} + 2) \\ = & 2^{\frac{2}{3}} \cdot \frac{1}{6} \cdot (a_i - a_{i+1} + 3). \end{aligned}$$

So

$$\begin{aligned} & \sum_{i=1}^{n} \sqrt[6]{a_i(1-a_{i+1})} \\ \leqslant & 2^{\frac{2}{3}} \cdot \frac{1}{6} \sum_{i=1}^{n} (a_i - a_{i+1} + 3) \\ = & 2^{\frac{2}{3}} \cdot \frac{1}{6} \cdot 3n \\ = & \frac{n}{\sqrt[3]{2}}. \end{aligned}$$

The equality holds if and only if $a_1 = a_2 = \cdots = a_n = \frac{1}{2}$. So

$$y_{\max} = \frac{n}{\sqrt[3]{2}}.$$

2 Find the smallest positive real number k such that for any four given distinct real numbers a, b, c and d, each greater than or equal to k, there exists a permutation p, q, r and s of a, b, c and d such that the equation

$$(x^2 + px + q)(x^2 + rx + s) = 0$$

has four distinct real roots.

Solution Suppose $k < 4$. Take $a, b, c, d \in [k, \sqrt{4k}]$. Then for any permutation p, q, r, s of a, b, c, d, consider the equation $x^2 + px + q = 0$, its discriminant

$$\Delta = p^2 - 4q < 4k - 4q \leqslant 4k - 4k = 0.$$

Therefore it has no real roots. So $k \geqslant 4$.

Suppose $4 \leqslant a < b < c < d$. Consider the following equations:

$$x^2 + dx + a = 0,$$

$$x^2 + cx + b = 0.$$

Observe that their discriminants

$$\Delta = d^2 - 4a > 4(d - a) > 0$$

and

$$\Delta = c^2 - 4b > 4(c - b) > 0.$$

Then the above two equations have two distinct real roots.

Suppose these two equations have the same real root β. Then we have

$$\beta^2 + d\beta + a = 0,$$

$$\beta^2 + c\beta + b = 0.$$

Taking their difference yields $\beta = \dfrac{b - a}{d - c} > 0$. Then

$\beta^2 + d\beta + a > 0$, which leads to a contradiction. So $k = 4$.

3 In $\triangle PBC$, $\angle PBC = 60°$. The tangent at point P to the circumcircle w of $\triangle PBC$ intersects with line CB at A. Points D and E lie on the line segment PA and circle w respectively, such that $\angle DBE = 90°$ and $PD = PE$. BE and PC meet at F. It is given that lines AF, BP and CD are concurrent.

(1) Prove that BF is the bisector of $\angle PBC$;

(2) Find the value of $\tan \angle PCB$.

Solution (1) When BF bisects $\angle PBC$, since $\angle DBE = 90°$, we know that BD is the bisector of $\angle PBA$.

By the angle bisector theorem, we have

$$\frac{PF}{FC} \cdot \frac{CB}{BA} \cdot \frac{AD}{DP} = \frac{PB}{BC} \cdot \frac{BC}{BA} \cdot \frac{AB}{PB} = 1.$$

By the converse of *Ceva* theorem, the lines AF, BP and CD are concurrent.

Suppose there exists $\angle D'BF'$ satisfying the conditions: (a) $\angle D'BF' = 90°$, (b) the lines AF', BP and CD' are concurrent. We may assume that F' lies on PF. Then, D' is on AD.

So

$$\frac{PF'}{F'C} < \frac{PF}{FC}, \frac{AD'}{PD'} < \frac{AD}{PD}.$$

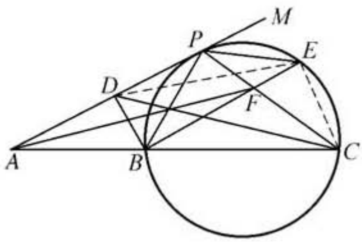
Thus

$$\frac{PF'}{F'C} \cdot \frac{CB}{BA} \cdot \frac{AD'}{D'P} < \frac{PB}{BC} \cdot \frac{BC}{BA} \cdot \frac{AB}{PB} = 1,$$

which leads to a contradiction. This completes the proof.

(2) We may assume that the circle O has radius 1. Let

$\angle PCB = \alpha$. By (1), $\angle PBE = \angle EBC = 30^\circ$. Therefore, E is the midpoint of $\overparen{PC}$.



Since $\angle MPE = \angle PBE = 30^\circ$, $\angle CPE = \angle CBE = 30^\circ$ and $PD = PE$, we obtain $\angle PDE = \angle PED = 15^\circ$, $PE = 2 \cdot 1 \cdot \sin 30^\circ$ and $DE = 2\cos 15^\circ$.

Since

$$BE = 2\sin \angle ECB = 2\sin(\alpha + 30^\circ)$$

and $\angle BED = \angle BEP - 15^\circ$, we have

$$\cos(\alpha - 15^\circ) = \frac{BE}{DE} = \frac{2\sin(\alpha + 30^\circ)}{2\cos 15^\circ},$$

$$\cos(\alpha - 15^\circ)\cos 15^\circ = \sin(\alpha + 30^\circ),$$

$$\cos\alpha + \cos(\alpha - 30^\circ) = 2\sin(\alpha + 30^\circ),$$

$$\cos\alpha + \cos\alpha\cos 30^\circ + \sin\alpha\sin 30^\circ = \sqrt{3}\sin\alpha + \cos\alpha,$$

$$1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\tan\alpha = \sqrt{3}\tan\alpha + 1.$$

So

$$\tan\alpha = \frac{6 + \sqrt{3}}{11}.$$

4 Assume that a is a positive integer and not a perfect square. Prove that for any positive integer n, the sum

$$S_n = \{\sqrt{a}\} + \{\sqrt{a}\}^2 + \cdots + \{\sqrt{a}\}^n$$

is irrational, where $\{x\} = x - [x]$ and $[x]$ denotes the greatest integer less than or equal to x.

Proof Suppose that $c^2 < a < (c+1)^2$, where c is an integer greater than or equal to 1. Then, $[\sqrt{a}] = c$, $1 \leqslant a - c^2 \leqslant 2c$, and $\{\sqrt{a}\} = \sqrt{a} - [\sqrt{a}] = \sqrt{a} - c$.

Write $\{\sqrt{a}\}^k = (\sqrt{a} - c)^k = x_k + y_k\sqrt{a}$, where $k \in \mathbf{N}$ and x_k, $y_k \in \mathbf{Z}$. We have

$$S_n = (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n)\sqrt{a}. \quad ①$$

We now prove that $T_n = \sum_{k=1}^{n} y_k \neq 0$ for all positive integers n.

Since

$$\begin{aligned} x_{k+1} + y_{k+1}\sqrt{a} &= (\sqrt{a} - c)^{k+1} \\ &= (\sqrt{a} - c)(x_k + y_k\sqrt{a}) \\ &= (ay_k - cx_k) + (x_k - cy_k)\sqrt{a}, \end{aligned}$$

we have

$$x_{k+1} = ay_k - cx_k, \ y_{k+1} = x_k - cy_k.$$

Since $x_1 = -c$ and $y_1 = 1$, we have $y_2 = -2c$.

By the above equality, we have

$$y_{k+2} = -2cy_{k+1} + (a - c^2)y_k, \quad ②$$

where $y_1 = 1$, $y_2 = -2c$.

By mathematical induction, we have

$$y_{2k-1} > 0, \ y_{2k} < 0. \quad ③$$

Combining ② and ③, we have

$$y_{2k+2} - y_{2k+1} = -(2c+1)y_{2k+1} + (a - c^2)y_{2k} < 0,$$

$$y_{2k+2} + y_{2k+1} = -(2c-1)y_{2k+1} + (a - c^2)y_{2k} < 0.$$

Taking the product of the above inequalities, we get $y_{2k+2}^2 - y_{2k+1}^2 > 0$. Since $y_2^2 - y_1^2 > 0$, we have

$$|y_{2k-1}| < |y_{2k}|. \quad ④$$

On the other hand, we have

$$y_{2k+1} - y_{2k} = -(2c+1)y_{2k} + (a-c^2)y_{2k-1} > 0,$$

$$y_{2k+1} + y_{2k} = -(2c-1)y_{2k} + (a-c^2)y_{2k-1} > 0.$$

Multiplying the above inequalities, we have $y_{2k+1}^2 - y_{2k}^2 > 0$, that is, $|y_{2k}| < |y_{2k+1}|$.

Therefore, $|y_n| < |y_{n+1}|$ for all positive integers n. Combining ③ and ④, we have $y_{2k-1} + y_{2k} < 0$, $y_{2k+1} + y_{2k} > 0$ for all positive integers n.

Therefore,

$$T_{2n-1} = y_1 + (y_2 + y_3) + \cdots + (y_{2n-2} + y_{2n-1}) > 0,$$

$$T_{2n} = (y_1 + y_2) + (y_3 + y_4) + \cdots + (y_{2n-1} + y_{2n}) < 0.$$

Hence, $T_n \neq 0$ for all positive integer n. This completes the proof.

Second Day

0800 - 1200 November 5, 2006

5 Let $S = \{n \mid n-1, n, n+1$ all can be expressed as the sum of the squares of two positive integers$\}$. Prove that, if $n \in S$, then $n^2 \in S$.

Proof Note that if x and y are integers, then we have

$$x^2 + y^2 \equiv 0, 1, 2 \pmod 4.$$

Let $n \in S$. By the above equality, we get $n \equiv 1 \pmod 4$. Thus, we may assume that

$$n - 1 = a^2 + b^2,\ a \geqslant b,$$

$$n = c^2 + d^2,\ c > d,$$

$$n + 1 = e^2 + f^2,\ e \geqslant f,$$

where a, b, c, d, e, f are positive integers. Therefore

$$n^2 + 1 = n^2 + 1^2,$$

$$n^2 = (c^2 + d^2)^2 = (c^2 - d^2)^2 + (2cd)^2,$$

$$n^2 - 1 = (a^2 + b^2)(e^2 + f^2) = (ae - bf)^2 + (af + be)^2.$$

Suppose that $b = a$ and $f = e$, we have $n - 1 = 2a^2$, $n + 1 = 2e^2$. Taking the difference of these two equations yields $e^2 - a^2 = 1$. Then, $e - a \geqslant 1$. But

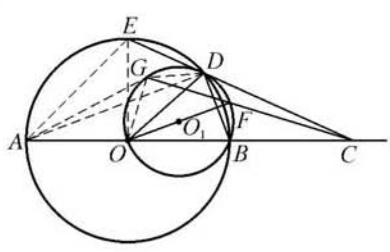
$$1 = e^2 - a^2 = (e + a)(e - a) > 1,$$

a contradiction!

So $b = a$ and $f = e$ do not happen simultaneously. Therefore $ae - bf > 0$, and $n^2 \in S$.

6 AB is a diameter of the circle O, the point C lies on the extend line AB produced. A line passing through C intersects with the circle O at points D and E. OF is a diameter of circumcircle O_1 of $\triangle BOD$. Join CF and its extension, intersects the circle O_1 at G. Prove that points O, A, E, G are concyclic.

Proof Because OF is a diameter of the circumcircle of $\triangle DOB$, OF is the bisector of $\angle DOB$, that's $\angle DOB = 2\angle DOF$. Since $\angle DAB = \frac{1}{2}\angle DOB$, we have $\angle DAB = \angle DOF$.



Since $\angle DGF = \angle DOF$, we obtain $\angle DGF = \angle DAB$.

Thus, G, A, C, D are concyclic. Hence,

$$\angle AGC = \angle ADC, \tag{①}$$

$$\angle AGC = \angle AGO + \angle OGF = \angle AGO + \frac{\pi}{2}, \tag{②}$$

$$\angle ADC = \angle ADB + \angle BDC = \angle BDC + \frac{\pi}{2}. \tag{③}$$

Combining ①, ② and ③ yields

$$\angle AGO = \angle BDC. \tag{④}$$

Since B, D, E, A are concyclic, we have

$$\angle BDC = \angle EAO. \tag{⑤}$$

As $OA = OE$, we obtain

$$\angle EAO = \angle AEO. \tag{⑥}$$

Combining ④, ⑤ and ⑥ implies $\angle AGO = \angle AEO$.

Therefore, O, A, E, G are concyclic.

7 Let k be a positive integer no less than 3 and θ be a real number. Prove that, if both $\cos(k-1)\theta$ and $\cos k\theta$ are rational numbers, then there exists a positive integer $n > k$, such that both $\cos(n-1)\theta$ and $\cos n\theta$ are rational numbers.

Proof First we prove a lemma.

Lemma *Let α be a real number. If $\cos\alpha$ is rational, then $\cos m\alpha$ is rational for any positive integer m.* $(*)$

We prove by induction on m. By $\cos 2\alpha = 2\cos^2\alpha - 1$, we get that $(*)$ is true for $m = 2$.

We suppose that $(*)$ is true for $m \leqslant l$ $(l \geqslant 2)$.

Since

$$\cos(l+1)\alpha = 2\cos l\alpha \cdot \cos\alpha - \cos(l-1)\alpha,$$

then we conclude that ($*$) is true for $m = l + 1$ and our induction is complete.

By the lemma, setting $m = k$, $m = k+1$ for $\alpha = k\theta$, $(k-1)\theta$, it follows that $\cos k^2\theta$, $\cos(k^2-1)\theta$ are rational numbers. Since $k^2 > k$, the statement holds.

8 Given a positive integer $n \geqslant 2$, let $B_1, B_2, \cdots, B_n$ denote n arbitrary subsets of set X, each of which contains exactly two elements. Find the minimum value of $|X|$ such that there exists a subset Y of set X satisfying:

(a) $|Y| = n$;

(b) $|Y \cap B_i| \leqslant 1$ for $i = 1, 2, \cdots, n$,

where $|A|$ denotes the number of elements of the finite set A.

Solution We first prove that $|X| > 2n-2$. In fact, if $|X| = 2n-2$, let $X = \{1, 2, \cdots, 2n-2\}$, $B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$, $\cdots$, $B_{n-1} = \{2n-3, 2n-2\}$. Since $|Y| = n$, there exist two elements in Y that belong to the same B_i, then $|Y \cap B_i| > 1$, a contradiction.

Let $|X| = 2n-1$.

Let $B = \bigcup_{i=1}^{n} B_i$, then $|B| = 2n-1-z$, where z is the number of subset $X \backslash B$. Suppose the elements of $X \backslash B$ are a_1, a_2, $\cdots$, a_z.

If $z \geqslant n-1$, take $Y = \{a_1, \cdots, a_{n-1}, d\}$, and $d \in B$, as desired.

If $z < n-1$, suppose there are t elements that occur once in $B_1, B_2, \cdots, B_n$. Since $\sum_{i=1}^{n} |B_i| = 2n$, then

$$t+2(2n-1-z-t)\leqslant 2n,$$

it follows that $t\geqslant 2n-2-2z$. So the elements that occur twice or more than twice in $B_1, B_2, \cdots, B_n$ occur repeatedly by $2n-(2n-2-2z)=2+2z$ times.

Consider the elements that occur once in $B_1, B_2, \cdots, B_n$: $b_1, b_2, \cdots, b_t$. Thus, there are at most $\frac{2+2z}{2}=1+z$ subsets in $B_1, B_2, \cdots, B_n$ that do not contain the elements $b_1, b_2, \cdots, b_t$. So, there exist $n-(z+1)=n-z-1$ subsets containing at least the elements $b_1, b_2, \cdots, b_t$.

Suppose that $B_1, B_2, \cdots, B_{n-1-z}$ contain the elements $\widetilde{b_1}, \widetilde{b_2}, \cdots, \widetilde{b}_{n-1-z}$ of $b_1, b_2, \cdots, b_t$, respectively. Since

$$2(n-1-z)+z=2n-2-z<2n-1,$$

there must exist an element d that is not in $B_1, B_2, \cdots, B_{n-1-z}$ but is in $B_{n-z}, \cdots, B_n$.

Write $Y=\{a_1, \cdots, a_z, \widetilde{b_1}, \widetilde{b_2}, \cdots, \widetilde{b}_{n-1-z}, d\}$, as desired.

2007 (Nanning, Guangxi)

The 7th (2007) China Western Mathematical Olympiad was held 8~13 November, 2007 in Nanning, Guangxi, China, and was hosted by Guangxi Mathematical Society and Nanning No. 2 high School.

The competition committee comprised: Xiong Bin, Wu Jianping, Chen Yonggao, Li Shenghong, Li Weigu, Wang Jianwei, Zhao Jiyuan, Liu Shixiong, Feng Zhigang, Bian

Hongping.

First Day

0800 - 1200 November 10, 2007

1 Let $T = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Find the number of all nonempty subsets A of T such that $3 \mid S(A)$ and $5 \nmid S(A)$, where $S(A)$ is the sum of all elements of A.

Solution Define $S(\varnothing) = 0$, Let $T_0 = \{3, 6\}$, $T_1 = \{1, 4, 7\}$, $T_2 = \{2, 5, 8\}$. For $A \subseteq T$, Let $A_0 = A \cap T_0$, $A_1 = A \cap T_1$, $A_2 = A \cap T_2$, then

$$\begin{aligned} S(A) &= S(A_0) + S(A_1) + S(A_2) \\ &\equiv |A_1| - |A_2| \pmod 3, \end{aligned}$$

So $3 \mid S(A)$ if and only if $|A_1| \equiv |A_2| \pmod 3$. It follows that

$$\begin{aligned} (|A_1|, |A_2|) = {} & (0, 0), (0, 3), (3, 0), \\ & (3, 3), (1, 1), (2, 2). \end{aligned}$$

The number of nonempty subsets A so that $3 \mid S(A)$ is

$$2^2\binom{3}{0}\binom{3}{0} + \binom{3}{0}\binom{3}{3} + \binom{3}{3}\binom{3}{0} + \binom{3}{3}\binom{3}{3}$$
$$+ \binom{3}{1}\binom{3}{1} + \binom{3}{2}\binom{3}{2} - 1 = 87.$$

If $3 \mid S(A)$ and $5 \mid S(A)$, then $15 \mid S(A)$. Since $S(T) = 36$, so the value $S(A)$ is 15 or 30 (If $3 \mid S(A)$ and $5 \mid S(A)$).

Furthermore,

$$\begin{aligned} 15 &= 8 + 7 = 8 + 6 + 1 = 8 + 5 + 2 = 8 + 4 + 3 \\ &= 8 + 4 + 2 + 1 = 7 + 6 + 2 = 7 + 5 + 3 \\ &= 7 + 5 + 2 + 1 = 7 + 4 + 3 + 1 = 6 + 5 + 4 \end{aligned}$$

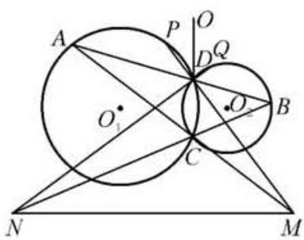
$$= 6 + 5 + 3 + 1 = 6 + 4 + 3 + 2$$
$$= 5 + 4 + 3 + 2 + 1,$$
$$36 - 30 = 6 = 5 + 1 = 4 + 2 = 3 + 2 + 1.$$

So the number of A such that $3 | S(A)$, $5 | S(A)$, and $A \neq \varnothing$ is 17.

The answer is $87 - 17 = 70$.

2 Let C and D be two intersection points of circle O_1 and circle O_2. A line, passing through D, intersects circle O_1 and circle O_2 at points A and B respectively. The points P and Q are on circle O_1 and circle O_2 respectively. The lines PD and AC intersect at M, and the lines QD and BC intersect at N. Suppose O is the circumcenter of the triangle ABC, prove that $OD \perp MN$ if and only if P, Q, M and N are concyclic.

Proof Let the circumcenter of the triangle ABC is O, and the radius of the circle O is R. Then



$$NO^2 - R^2 = NC \cdot NB, \quad ①$$
$$MO^2 - R^2 = MC \cdot MA. \quad ②$$

Since A, C, D and P are concyclic, so we have

$$MC \cdot MA = MD \cdot MP. \quad ③$$

Similarly, since Q, D, C and B are concyclic, we have

$$NC \cdot NB = ND \cdot NQ. \quad ④$$

From ①, ②, ③, ④, we have

$$NO^2 - MO^2 = ND \cdot NQ - MD \cdot MP$$
$$= ND(ND + DQ) - MD(MD + DP)$$

$$= ND^2 - MD^2 + ND \cdot DQ - MD \cdot DP$$

So,

$$\begin{aligned} OD \perp MN &\Leftrightarrow NO^2 - MO^2 = ND^2 - MD^2 \\ &\Leftrightarrow ND \cdot DQ = MD \cdot DP \\ &\Leftrightarrow P, Q, M, N \text{ are concyclic.} \end{aligned}$$

3 Suppose a, b, c are real numbers, with $a + b + c = 3$. Prove that

$$\frac{1}{5a^2 - 4a + 11} + \frac{1}{5b^2 - 4b + 11} + \frac{1}{5c^2 - 4c + 11} \leqslant \frac{1}{4}.$$

Proof If $a < \frac{9}{5}$, then

$$\frac{1}{5a^2 - 4a + 11} \leqslant \frac{1}{24}(3 - a). \qquad ①$$

In fact,

$$\begin{aligned} ① &\Leftrightarrow (3 - a)(5a^2 - 4a + 11) \geqslant 24 \\ &\Leftrightarrow 5a^3 - 19a^2 + 23a - 9 \leqslant 0 \\ &\Leftrightarrow (a - 1)^2(5a - 9) \leqslant 0 \Leftarrow a < \frac{9}{5}. \end{aligned}$$

So if a, b, $c < \frac{9}{5}$, then

$$\begin{aligned} &\frac{1}{5a^2 - 4a + 11} + \frac{1}{5b^2 - 4b + 11} + \frac{1}{5c^2 - 4c + 11} \\ \leqslant &\frac{1}{24}(3 - a) + \frac{1}{24}(3 - b) + \frac{1}{24}(3 - c) \\ = &\frac{1}{4}. \end{aligned}$$

If one of a, b, c is not less than $\frac{9}{5}$, say $a \geqslant \frac{9}{5}$, then

$$\begin{aligned}5a^2-4a+11&=5a\left(a-\frac{4}{5}\right)+11\\&\geqslant 5\cdot\frac{9}{5}\cdot\left(\frac{9}{5}-\frac{4}{5}\right)+11=20.\end{aligned}$$

So $\frac{1}{5a^2-4a+11}\leqslant\frac{1}{20}$.

Since

$$5b^2-4b+11=5\left(b-\frac{2}{5}\right)^2+11-\frac{4}{5}\geqslant 11-\frac{4}{5}>10,$$

we have $\frac{1}{5b^2-4b+11}<\frac{1}{10}$. Similarly, $\frac{1}{5c^2-4c+11}<\frac{1}{10}$. So

$$\begin{aligned}&\frac{1}{5a^2-4a+11}+\frac{1}{5b^2-4b+11}+\frac{1}{5c^2-4c+11}\\&<\frac{1}{20}+\frac{1}{10}+\frac{1}{10}=\frac{1}{4}.\end{aligned}$$

Hence the inequality is holds for all a, b, c.

4 Let O be an interior point of the triangle ABC. Prove that there exist positive integers p, q and r, such that

$$|p\cdot\overrightarrow{OA}+q\cdot\overrightarrow{OB}+r\cdot\overrightarrow{OC}|<\frac{1}{2007}.$$

Proof It is well-known that there are positive real numbers β, γ such that

$$\overrightarrow{OA}+\beta\overrightarrow{OB}+\gamma\overrightarrow{OC}=\vec{0}.$$

So for positive integer k, we have

$$k\overrightarrow{OA}+k\beta\overrightarrow{OB}+k\gamma\overrightarrow{OC}=\vec{0}.$$

Let $m(k)=[k\beta]$, $n(k)=[k\gamma]$, where $[x]$ is the biggest integer which is less than or equal to x, and $\{x\}=x-[x]$.

Assume T is an integer larger than $\max\left\{\frac{1}{\beta}, \frac{1}{\gamma}\right\}$. Then the sequences $\{m(kT) \mid k = 1, 2, \cdots\}$ and $\{n(kT) \mid k = 1, 2, \cdots\}$ are increasing, and

$$\begin{aligned}
&|kT\overrightarrow{OA} + m(kT)\overrightarrow{OB} + n(kT)\overrightarrow{OC}| \\
= &|-\{kT\beta\}\overrightarrow{OB} - \{kT\gamma\}\overrightarrow{OC}| \\
\leqslant &|\overrightarrow{OB}| \cdot \{kT\beta\} + |\overrightarrow{OC}| \cdot \{kT\gamma\} \\
\leqslant &|\overrightarrow{OB}| + |\overrightarrow{OC}|.
\end{aligned}$$

This shows there exists infinitely many vectors such that

$$kT\overrightarrow{OA} + m(kT)\overrightarrow{OB} + n(kT)\overrightarrow{OC},$$

whose endpoint is in a circle O of radius $|\overrightarrow{OB}| + |\overrightarrow{OC}|$. So there are two of the vectors, such that the distance between the endpoints of the two vectors is less than $\frac{1}{2007}$, this means there exists two integers $k_1 < k_2$, such that

$$\begin{aligned}
&|(k_2T\overrightarrow{OA} + m(k_2T)\overrightarrow{OB} + n(k_2T)\overrightarrow{OC}) \\
&\quad - (k_1T\overrightarrow{OA} + m(k_1T)\overrightarrow{OB} + n(k_1T)\overrightarrow{OC})| \\
< &\frac{1}{2007}.
\end{aligned}$$

So, if we let $p = (k_2 - k_1)T$, $q = m(k_2T) - m(k_1T)$, $r = n(k_2T) - n(k_1T)$, then p, q, r are integers, and

$$|p\overrightarrow{OA} + q\overrightarrow{OB} + r\overrightarrow{OC}| < \frac{1}{2007}.$$

Second Day

0800 - 1200 November 11, 2007

5 Is there a triangle with sides of integral length, such that the length of the shortest side is 2007 and that the largest angle is twice the smallest?

Solution We shall prove that no such a triangle satisfies the condition.

If $\triangle ABC$ satisfies the condition, let $\angle A \leqslant \angle B \leqslant \angle C$, then $\angle C = 2\angle A$, and $a = 2007$. Draw the bisector of $\angle ACB$ intersets AB at point D. Then $\angle BCD = \angle A$, so $\triangle CDB \backsim \triangle ACB$, it follows that

$$\frac{CB}{AB} = \frac{BD}{BC} = \frac{CD}{AC} = \frac{BD + CD}{BC + AC} = \frac{AB}{BC + AC}.$$

Thus

$$c^2 = a(a + b) = 2007(2007 + b), \qquad ①$$

where $2007 \leqslant b \leqslant c < 2007 + b$.

Since a, b, c are integers, so $2007 \mid c^2$, then $3 \cdot 223 \mid c^2$. We can let $c = 669m$, from ①, we get $223m^2 = 2007 + b$. Thus $b = 223m^2 - 2007 \geqslant 2007$, so $m \geqslant 5$.

But $c \geqslant b$, so $669m \geqslant 223m^2 - 2007$, this implies $m < 5$, contradiction.

6 Find all positive integers n such that there exist non-zero integers x_1, x_2, $\cdots$, x_n, y, satisfying the following conditions

$$\begin{cases} x_1 + \cdots + x_n = 0, \\ x_1^2 + \cdots + x_n^2 = ny^2. \end{cases}$$

Solution It is easy to see that $n > 1$.

When $n = 2k$, $k \in \mathbf{N}$, let $x_{2i-1} = 1$, $x_{2i} = -1$, $i = 1$, 2, $\cdots$, k, and $y = 1$, then the condition is satisfied.

When $n = 2k + 3$, $k \in \mathbf{N}$, let $y = 2$, $x_1 = 4$,

$x_2 = \cdots = x_5 = -1$, $x_{2i} = 2$, $x_{2i+1} = -2$, $i = 3, 4, \cdots, k+1$, then the condition is satisfied.

Now if $n = 3$, and there exist x_1, x_2, x_3, y such that

$$\begin{cases} x_1 + x_2 + x_3 = 0, \\ x_1^2 + x_2^2 + x_3^2 = 3y^2. \end{cases}$$

then

$$2(x_1^2 + x_2^2 + x_1x_2) = 3y^2.$$

Suppose gcd $(x_1, x_2) = 1$, so x_1, x_2 are odd numbers or one is even while the other is odd. Hence $x_1^2 + x_2^2 + x_1x_2$ is an odd number. But $2 \mid 3y^2$, so $3y^2 \equiv 0 \pmod 4$. This is a contradiction since $2(x_1^2 + x_2^2 + x_1x_2) \equiv 2 \pmod 4$.

The answer is $n \neq 1, 3$.

7 Let P be an interior point of an acute-angled triangle ABC. The lines AP, BP, CP meet BC, CA, AB at D, E, F respectively. Given that $\triangle DEF \backsim \triangle ABC$. Prove that P is the centroid of $\triangle ABC$.

Proof Identify the points as complex numbers, with the origin O coinciding with M. Because M is interior to $\triangle ABC$, there are positive real numbers α, β, γ with $\alpha A + \beta B + \gamma C = 0$ and

$$\alpha + \beta + \gamma = 1. \qquad ①$$

Since D is on BC and on line OA $(= MA)$, it follows that $D = \frac{-\alpha}{1-\alpha}A$. Similarly, $E = \frac{-\beta}{1-\beta}B$ and $F = \frac{-\gamma}{1-\gamma}C$.

Because $\triangle ABC$ is similar to $\triangle DEF$, we have

$$\frac{D-E}{A-B} = \frac{E-F}{B-C}.$$

Substituting our expressions for D, E and F yields

$$\frac{\gamma BC}{1-\gamma}+\frac{\beta AB}{1-\beta}+\frac{\alpha AC}{1-\alpha}-\frac{\alpha AB}{1-\alpha}-\frac{\beta BC}{1-\beta}-\frac{\gamma AC}{1-\gamma}=0. \qquad ②$$

Taking ① into account, equation ② implies that

$$BC(\gamma^2-\beta^2)+CA(\alpha^2-\gamma^2)+AB(\beta^2-\alpha^2)=0,$$

and that

$$(\gamma^2-\beta^2)B(C-A)+(\alpha^2-\gamma^2)A(C-B)=0.$$

If $\gamma^2-\beta^2\neq 0$, then the cross-ratio

$$\frac{(C-A)/(C-B)}{(M-A)/(M-B)}=\frac{B(C-A)}{A(C-B)}$$

is a real number. But this implies that M is on the circumcircle of $\triangle ABC$, contradicting the fact that M is in the interior of the triangle. Thus $\gamma^2=\beta^2$, and $\alpha^2=\gamma^2$ as well. It follows that $\alpha=\beta=\gamma=\frac{1}{3}$, and hence, M is the centroid of $\triangle ABC$.

8 There are n white and n black balls placed randomly on the circumference of a circle. Starting from a certain white ball, number all white balls in a clockwise direction by $1, 2, \cdots, n$. Likewise, number all black balls by $1, 2, \cdots, n$ in an anti-clockwise direction starting from a certain black ball. Prove that there exist consecutive n balls whose numbering forms the set $\{1, 2, \cdots, n\}$.

Proof Choose a black ball and a white ball with the same number, and the number of balls between the two balls is minimum. We can suppose the number of the two balls is 1.

Firstly, we shall prove that the balls between the two balls have the same color.

In fact, if they are of different color, then the white ball and the black ball, each is numbered by n, are between the two ball (See Fig. 1). This is a contradiction to the point that number of balls between the two balls (labelled by '1' s) is a minimum.

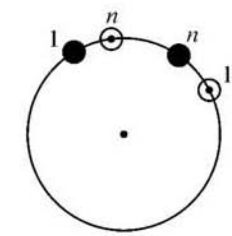


Fig. 1

Secondly, if the balls between the two balls (labelled by '1's) are white, we have two cases.

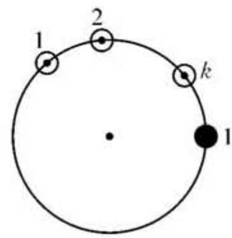


Fig. 2

Case 1 The number of the white balls are 2, $\cdots$, k. See Fig. 2, then from the white balls (labelled by '1' s) in anti-clockwise direction we can get a chain of n balls whose numbering forms the set $\{1, 2, \cdots, n\}$.

Case 2 The number of the white balls are k, $k+1$, $\cdots$, n (See Fig. 3), then from the white ball (labelled by '1' s) in clockwise direction we can have a chain of n balls that satisfies the condition.

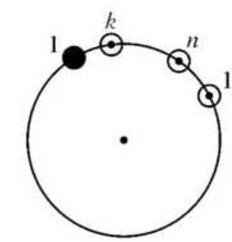


Fig. 3

The same argument can prove that the claim holds, if the balls between the two balls (labelled by '1's) are black, or there are no ball between them.

China Southeastern Mathematical Olympiad

2007 (Zhenhai, Zhejiang)

The fourth (2007) China Southeastern Mathematical Olympiad was held from 25 to 30 July, 2007 in Zhenhai, Zhejiang, China, and was hosted by the Southeastern Mathematical Society and Zhenhai High School. About 200 students of grade 10, from Fujian, Zhejiang and Jiangxi province, formed 39 teams to take part in the competition. Apart from these three provinces, teams from Hongkong, Shanghai and Guangdong were also invited to participate in the competition. The purpose of this competition is to provide students with a platform for an exchange of Mathematical Olympiad Problems. Besides the competition, there were also lectures delivered by experts and activities for teachers and students to exchange ideas.

The Competition Committee comprises the following: Tao Pingsheng, Li Shenghong, Zhang Pengcheng, Jin Mengwei and Yang Xiaoming.

First Day

0800 - 1200 July 27, 2007

1. How many integers a satisfy the condition: for each a, the equation $x^3 = ax + a + 1$ with respect to x has roots which are even and $|x| < 1000$.

Solution I Let $x_0 = 2n$, where n is an integer and $|2n| < 1000$, then $|n| \leqslant 499$. So we can choose at most $2 \times 499 + 1 = 999$ numbers, that is, $n \in \{-499, -498, \cdots, 0, 1, \cdots, 499\}$. Substituting $x_0 = 2n$ into the equation, we get $a = \dfrac{8n^3 - 1}{2n + 1}$.

Set $f(n) = \dfrac{8n^3 - 1}{2n + 1}$, then for any $n_1, n_2 \in \{-499, -498, \cdots, 0, 1, \cdots, 499\}$ $(n_1 \neq n_2, n_1, n_2 \in \mathbf{Z})$. If $f(n_1) = f(n_2)$, we can suppose $n_1 = \dfrac{x_1}{2}$, $n_2 = \dfrac{x_2}{2}$, where x_1, x_2 are roots of $x^3 - ax - a - 1 = 0$. Suppose the other root is x_3, then according to the sums and products of roots, we obtain

$$\begin{cases} x_3 = -(x_1 + x_2), \\ x_1x_2 + x_2x_3 + x_3x_1 = -a, \\ x_1x_2x_3 = a + 1, \end{cases}$$

that is,

$$\begin{cases} 4N_1 = -a, \\ 8N_2 = a+1, \end{cases}$$

where $N_1 = -(n_1^2 + n_2^2 + n_1 n_2)$, $N_2 = -n_1 n_2 (n_1 + n_2)$, then $4N_1 + 8N_2 = 1$. Contradiction!

Thus, for any n_1, $n_2 \in \{-499, -498, \cdots, 0, 1, \cdots, 499\}$, we have $f(n_1) \neq f(n_2)$, which means there are exactly 999 real numbers a that satisfy the condition.

Solution Ⅱ Our aim is to prove that for any even integer x which satisfies $|x| \leqslant 998$, the value of $a = \dfrac{x^3 - 1}{x+1}$ is different.

On the contrary, if there exist $x_1 \neq x_2$ that satisfy $\dfrac{x_1^3 - 1}{x_1 + 1} = \dfrac{x_2^3 - 1}{x_2 + 1}$, where x_1, x_2 are even numbers, then

$$(x_1 - x_2)(x_1^2 x_2 + x_1 x_2^2 + x_1^2 + x_2^2 + x_1 x_2 + 1) = 0.$$

Since $x_1 \neq x_2 \Rightarrow x_1 - x_2 \neq 0$ and $x_1^2 x_2 + x_1 x_2^2 + x_1^2 + x_2^2 + x_1 x_2$ is an even number, we obtain

$$(x_1 - x_2)(x_1^2 x_2 + x_1 x_2^2 + x_1^2 + x_2^2 + x_1 x_2 + 1) \neq 0.$$

Contradiction!

Thus, there exist 999 real numbers a that satisfy the condition.

2 As shown in the figure, points C and D are on the semicircle with O as its center and AB as its diameter. The tangent line to the semicircle at point B meets the line CD at point P. Line PO intersects CA and AD at point E and F respectively. Prove: $OE = OF$.

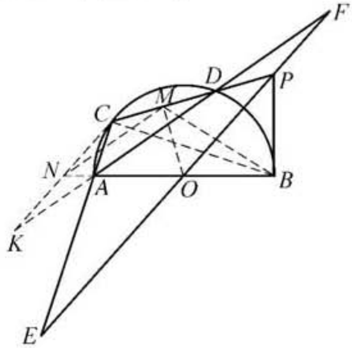
Proof Ⅰ Draw line segments OM and MN, such that $OM \perp CD$, $MN \mathbin{/\mkern-6mu/} AD$. Let $MN \cap BA = N$, $CN \cap DA = K$, and connect BC and BM, then we get

$$\angle NBC = \angle ADC = \angle NMC,$$

which means points N, B, M, C are concyclic. Since O, B, P, M are also concyclic points, we obtain

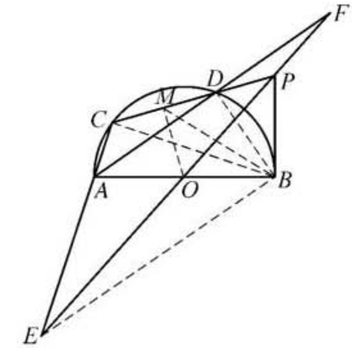
$$\angle OPM = \angle OBM = 180^\circ - \angle MCN,$$

thus $CN \parallel OP$ and



$$\frac{CN}{OE} = \frac{AN}{AO} = \frac{NK}{OF}. \qquad ①$$

Since M is the midpoint of CD, $MN \parallel DK$, then we get that N is the midpoint of CK. Hence, according to ①, we have $OE = OF$ as desired.



Proof Ⅱ As shown in the figure, draw $OM \perp CD$ with M as the foot of the perpendicular, and connect BC, BM, BD and BE. Since $OM \perp CD$, $PB \perp AB$, then O, B, P, M are concyclic points. Hence,

$$\begin{aligned}
&\angle BMP = \angle BOP = \angle AOE,\ \angle EAO = \angle BDM \\
&\quad \Rightarrow \triangle OAE \backsim \triangle MDB,\ \frac{AE}{BD} = \frac{AO}{DM} = \frac{AB}{CD} \\
&\quad \Rightarrow \triangle BAE \backsim \triangle CDB,\ \angle EBA = \angle BCD = \angle BAD \\
&\quad \Rightarrow AD \parallel BE,\ \frac{OE}{OF} = \frac{OB}{OA} = 1.
\end{aligned}$$

Therefore, we have $OE = OF$ as desired.

3 Suppose $a_i = \min\left\{k + \frac{i}{k} \mid k \in \mathbf{N}^*\right\}$, find the value of

$S_{n^2} = [a_1] + [a_2] + \cdots + [a_{n^2}]$, where $n \geqslant 2$, and $[x]$ denotes the greatest integer less than or equal to x.

Solution Set

$$a_{i+1} = \min\left\{k + \frac{i+1}{k} \mid k \in \mathbf{N}^*\right\} = k_1 + \frac{i+1}{k_1} \ (k_1 \in \mathbf{N}^*),$$

then

$$a_i \leqslant k_1 + \frac{i}{k_1} < k_1 + \frac{i+1}{k_1} = a_{i+1},$$

which means $\{a_n\}$ is a monotonic increasing sequence. Since $k + \frac{m^2}{k} \geqslant 2m$ (where the equality holds if and only if $k = m$), we have $a_{m^2} = 2m (m \in \mathbf{N}^*)$.

On the other hand,

$$k + \frac{m(m+1)}{k} = 2m + 1$$

when $k = m$, $m + 1$. But when $k \leqslant m$ or $k \geqslant m + 1$, we get $(k - m)(k - m - 1) \geqslant 0$, which means

$$k^2 - (2m+1)k + m(m+1) \geqslant 0$$
$$\Rightarrow k + \frac{m(m+1)}{k} \geqslant 2m + 1.$$

Thus, we have $a_{m^2+m} = 2m + 1$. Moreover, with regards to the monotonicity of $\{a_n\}$, we have $2m + 1 \leqslant a_i < 2(m+1)$ when $m^2 + m \leqslant i < (m+1)^2$. Hence,

$$[a_i] = \begin{cases} 2m, & m^2 \leqslant i < m^2 + m, \\ 2m+1, & m^2 + m \leqslant i < (m+1)^2. \end{cases}$$

Therefore,

$$\sum_{i=m^2}^{m^2+2m} [a_i] = 2m \cdot m + (2m+1) \cdot (m+1)$$
$$= 4m^2 + 3m + 1,$$

and

$$\begin{aligned} S_{n^2} &= \sum_{m=1}^{n-1}(4m^2+3m+1)+2n \\ &= 4\times\frac{n(n-1)(2n-1)}{6}+3\times\frac{n(n-1)}{2}+(n-1)+2n \\ &= \frac{8n^3-3n^2+13n-6}{6}. \end{aligned}$$

4 Find the smallest positive integer n such that any sequence of positive integers $a_1, a_2, \cdots, a_n$ satisfying $\sum_{i=1}^{n} a_i = 2007$ must have several consecutive terms whose sum is 30.

Solution Firstly, we could construct a sequence of positive integers with 1017 terms $a_1, a_2, \cdots, a_{1017}$, such that we cannot find consecutive terms whose sum is 30. Hence, we could set $a_1 = a_2 = \cdots = a_{29} = 1$, $a_{30} = 31$ and $a_{30m+i} = a_i$, $i \in \{1, 2, \cdots, 30\}$, $m \in \mathbf{N}$, which means the sequence is 1, 1, $\cdots$, 1, 31, | 1, 1, $\cdots$, 1, 31, |, $\cdots$, | 1, 1, $\cdots$, 1, 31, | 1, 1, $\cdots$, 1(in which there are 34 groups all have 30 terms except the last group with 27 terms, totalling 1017 terms).

Secondly, when the terms are less than 1017, what we should do is just to combine several consecutive terms into a larger number within certain groups of the sequence.

Now, for any sequence with 1018 terms $a_1, a_2, \cdots, a_{1018}$ that satisfies the condition $\sum_{i=1}^{1018} a_i = 2007$, we want to prove that there must exist several consecutive terms whose sum is 30. Denote $S_k = \sum_{i=1}^{k} a_i$, $k = 1, 2, \cdots, 1018$, then

$$1 \leqslant S_1 < S_2 < \cdots < S_{1018} = 2007.$$

Group the elements in the set $\{1, 2, \cdots, 2007\}$ as follows:

$(1, 31)$, $(2, 32)$, $\cdots$, $(30, 60)$;

$(61, 91)$, $(62, 92)$, $\cdots$, $(90, 120)$;

$(121, 151)$, $(122, 152)$, $\cdots$, $(150, 180)$;

…… …… …… …… ……

$(60k+1, 60k+31)$, $(60k+2, 60k+32)$, $\cdots$, $(60k+30, 60(k+1))$;

…… …… …… …… ……

$(60 \cdot 32+1, 60 \cdot 32+31)$, $(60 \cdot 32+2, 60 \cdot 32+32)$, $\cdots$, $(60 \cdot 32+30, 60 \cdot 33)$;

$1981, 1982, \cdots, 2007.$

There are $33 \times 30 = 990$ brackets and 27 numbers without brackets. Arbitrarily take 1018 numbers, whose sum is the value of S_k. There must be two numbers from the same bracket. Denote the two numbers by (S_k, S_{k+m}), then $S_{k+m} - S_k = 30$, which means,

$$a_{k+1} + a_{k+2} + \cdots + a_{k+m} = 30.$$

Therefore, the minimum of n is 1018.

Second Day

0800 – 1200 July 28, 2007

5 Let $f(x)$: $f(x+1) - f(x) = 2x + 1 (x \in \mathbf{R})$, and $|f(x)| \leqslant 1$ when $x \in [0, 1]$. Prove:

$$|f(x)| \leqslant 2 + x^2 (x \in \mathbf{R}).$$

Proof Let $g(x) = f(x) - x^2$, then

$$\begin{aligned} g(x+1) - g(x) &= f(x+1) - f(x) - (x+1)^2 + x^2 \\ &= 0. \end{aligned}$$

Thus, $g(x)$ is a periodic function with 1 as its period. On the other hand, as is given $|f(x)| \leqslant 1$ when $x \in [0, 1]$, so

$$|g(x)| = |f(x) - x^2| \leqslant |f(x)| + |x^2| \leqslant 2,$$

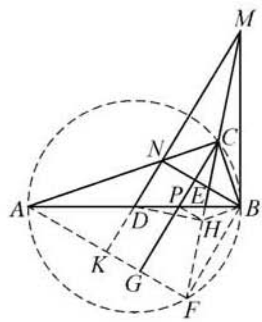
when $x \in [0, 1]$.

Therefore, the periodic function $g(x)$ satisfies $|g(x)| \leqslant 2 (x \in \mathbf{R})$, thus arriving at

$$\begin{aligned} |f(x)| &= |g(x) + x^2| \leqslant |g(x)| + |x^2| \\ &\leqslant 2 + x^2 (x \in \mathbf{R}), \end{aligned}$$

as desired.

6 As shown in the figure, in the right-angle triangle ABC, D is the midpoint of the hypotenuse AB, $MB \perp AB$, MD intersects AC at N, and MC is extended to intersect AB at E. Prove: $\angle DBN = \angle BCE$.



Proof Extend ME to intersect the circumscribed circle of $\triangle ABC$ at F, and MD to intersect AF at K. Draw $CG \mathbin{/\mkern-6mu/} MK$, which meet AF, AB at G and P respectively. On the other hand, draw $DH \perp CF$ with H as the foot of the perpendicular, then H is the midpoint of CF. Connect HB, HP, then D, H, B, M are concyclic, that is,

$$\angle HBD = \angle HMD = \angle HCP.$$

Hence, H, B, C, P are also concyclic, which means

$$\angle PHC = \angle ABC = \angle AFC$$

and $PH \mathbin{/\mkern-6mu/} AF$. Therefore, PH is the midline of $\triangle CFG$ and P

is the midpoint of CG, then AP is the median to CG in $\triangle ACG$.

Moreover, from $NK \parallel CG$, we get that D is the midpoint of NK. In other words, AB and NK bisect each other. Therefore, $\angle DBN = \angle DAK$, and

$$\angle DAK = \angle BAF = \angle BCF = \angle BCE,$$

then we prove $\angle DBN = \angle BCE$ as desired.

7 Find the array of prime numbers (a, b, c) satisfying conditions as follows:

(1) $a < b < c < 100$, where a, b, c are all prime numbers;

(2) $a + 1$, $b + 1$, $c + 1$ constitute the geometric progression.

Solution From condition (2), we get

$$(a+1)(c+1) = (b+1)^2. \quad ①$$

Set $a+1 = n^2 x$, $c+1 = m^2 y$, with no square factor larger than 1 in x, y, then we could get that $x = y$. This is due to the fact that from ①, we have

$$(mn)^2 xy = (b+1)^2, \quad ②$$

which means $mn \mid (b+1)$. Set $b+1 = mn \cdot w$, then ② can be simplified to

$$xy = w^2. \quad ③$$

If $w > 1$, then the prime number $p_1 \mid w \Rightarrow p_1^2 \mid w^2$. As there is no square factor larger than 1 in x, y, then $p_1 \mid x$ and $p_1 \mid y$. Now set $x = p_1 x_1$, $y = p_1 y_1$, $w = p_1 w_1$. Then ③ can be simplified to

$$x_1 y_1 = w_1^2. \quad ④$$

If $w_1 > 1$ still exists, then there will be a new prime number

$p_2 \mid w_1 \Rightarrow p_2^2 \mid w_1^2$. As there is no square factor larger than 1 in x_1, y_1, then $p_2 | x_1$ and $p_2 | y_1$. Now set

$$x_1 = p_2 x_2,\ y_1 = p_2 y_2,\ w_1 = p_2 w_2.$$

Then ④ can be simplified to $x_2 y_2 = w_2^2$, $\cdots$. Since there are finite prime factors of w in ③, then carrying on as above, we obtain that there exists r, such that $w_r = 1$. $x_r y_r = w_r^2 \Rightarrow x_r = y_r = 1$, we have $x = p_1 p_2 \cdots p_r = y$ as desired. Now we can set $x = y = k$, and have

$$\begin{cases} a = kn^2 - 1, \\ b = kmn - 1, \\ c = km^2 - 1, \end{cases} \quad ⑤$$

where

$$1 \leqslant n < m,\ a < b < c < 100, \quad ⑥$$

with no square factor larger than 1 in k and $k \neq 1$. Otherwise, if $k = 1$, then $c = m^2 - 1$. As c is larger than the third prime number 5, thus $c = m^2 - 1 > 5 \Rightarrow m \geqslant 3$ and

$$c = m^2 - 1 = (m-1)(m+1)$$

is a composite number. Contradiction! Hence, k is either a prime number, or the product of several different prime numbers (that is, k is larger than 1 and with no square factor larger than 1 in k). We say that "k has the property p".

(a) From ⑥, $m \geqslant 2$. When $m = 2$, then $n = 1$ and $\begin{cases} a = k - 1, \\ b = 2k - 1, \\ c = 4k - 1. \end{cases}$ Since $c < 100 \Rightarrow k < 25$, then if $k \equiv 1 \pmod 3$, we get $3 | c$ and $c > 3$, which means c is a composite number.

If $k \equiv 2 \pmod 3$, then when it is even, the k satisfying the property p is 2 or 14, where the corresponding $a=2-1=1$ and $b=2\cdot 14-1=27$ are not prime numbers. On the other hand, when it is odd, the k satisfying the property p is 5, 11, 17 or 23, where all the corresponding $a=k-1$ are not prime numbers.

If $k\equiv 0 \pmod 3$, the k satisfying the property p is 3, 6, 15 or 21. When $k=3$, we get the first solution

$$f_1=(a, b, c)=(2, 5, 11).$$

When $k=6$, the second solution is

$$f_2=(a, b, c)=(5, 11, 23).$$

But when $k=15$, 21, the corresponding $a=k-1$ are not prime numbers.

(b) When $m=3$, then $n=2$ or 1. If $m=3$, $n=2$, we have $\begin{cases} a=4k-1, \\ b=6k-1, \\ c=9k-1. \end{cases}$ Since $c\leqslant 97 \Rightarrow k\leqslant 10$, then the k satisfying the property p is 2, 3, 5, 6, 7 or 10.

When $k=3$, 5, 7, the corresponding $c=9k-1$ are all composite numbers.

When $k=6$, $b=6k-1=35$, which is a composite number.

When $k=10$, $a=4k-1=39$, which is also a composite number. But when $k=2$, we get the third solution

$$f_3=(a, b, c)=(7, 11, 17).$$

If $m=3$, $n=1$, we have $\begin{cases} a=k-1, \\ b=3k-1, \\ c=9k-1. \end{cases}$ As $k\leqslant 10$, the k satisfying the property p is 2, 3, 5, 6, 7 or 10. When $k=3$, 5, 7, the corresponding $b = 3k - 1$ are all composite numbers. When $k = 2, 10$, the corresponding $a = k - 1$ are not prime numbers. But when $k = 6$, we get the fourth solution

$$f_4 = (a, b, c) = (5, 17, 53).$$

(c) When $m = 4$, from $c = 16k - 1 \leqslant 97 \Rightarrow k \leqslant 6$, then the k satisfying the property p is 2, 3, 5 or 6. When $k = 6$, $c = 16 \cdot 6 - 1 = 95$, which is a composite number. When $k = 5$, then $\begin{cases} a = 5n^2 - 1, \\ b = 20n - 1. \end{cases}$ As $n < m = 4$, n can be 1, 2, 3, which means at least one of a, b is not a prime number.

When $k = 3$, $c = 48 - 1 = 47$ and $\begin{cases} a = 3n^2 - 1, \\ b = 12n - 1. \end{cases}$ Considering $n < m = 4$, the corresponding a, b are both composite numbers if $n = 3$. But, if $n = 2$, we get the fifth solution

$$f_5 = (a, b, c) = (11, 23, 47).$$

If $n = 1$, we get the sixth solution

$$f_6 = (a, b, c) = (2, 11, 47).$$

When $k = 2$, $c = 16 \cdot k - 1 = 31$ and $\begin{cases} a = 2n^2 - 1, \\ b = 8n - 1. \end{cases}$ Since $n < m = 4$, the seventh solution

$$f_7 = (a, b, c) = (17, 23, 31)$$

exists if $n = 3$.

(d) When $m = 5$, then $c = 25k - 1 \leqslant 97$ and the k satisfying the property p is 2 or 3, but the corresponding $c = 25k - 1$ are both composite numbers.

(e) When $m = 6$, then $c = 36k - 1 \leqslant 97$ and the k satisfying the property p is 2. Hence, $c = 2 \cdot 36 - 1 = 71$ and $\begin{cases} a = 2n^2 - 1, \\ b = 12n - 1. \end{cases}$ As $n < m = 6$, the eighth solution

$$f_8 = (a, b, c) = (7, 23, 71)$$

exists if $n = 2$, and the ninth solution

$$f_9 = (a, b, c) = (31, 47, 71)$$

exists if $n = 4$.

(f) When $m = 7$, then $c = 49k - 1 \leqslant 97$ and the k satisfying the property p is 2. Hence, $c = 2 \cdot 49 - 1 = 97$ and $\begin{cases} a = 2n^2 - 1, \\ b = 14n - 1. \end{cases}$ As $n < m = 7$, the tenth solution

$$f_{10} = (a, b, c) = (17, 41, 97)$$

exists if $n = 3$, and the eleventh solution

$$f_{11} = (a, b, c) = (71, 83, 97)$$

exists if $n = 6$.

(g) When $m \geqslant 8$, then $c = 64k - 1 \leqslant 97$, but the k satisfying the property p does not exist.

Therefore, there are 11 possible solutions, namely $f_1, f_2, \cdots, f_{11}$.

8 Given real numbers a, b, c such that $abc = 1$, prove that for all the integers $k \geqslant 2$,

$$\frac{a^k}{a+b} + \frac{b^k}{b+c} + \frac{c^k}{c+a} \geqslant \frac{3}{2}.$$

Proof Since

$$\frac{a^k}{a+b} + \frac{1}{4}(a+b) + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{k-2} \geqslant k \cdot \sqrt[k]{\frac{a^k}{2^k}} = \frac{k}{2}a,$$

then

$$\frac{a^k}{a+b} \geqslant \frac{k}{2}a - \frac{1}{4}(a+b) - \frac{k-2}{2}.$$

Similarly,

$$\frac{b^k}{b+c} \geqslant \frac{k}{2}b - \frac{1}{4}(b+c) - \frac{k-2}{2},$$

$$\frac{c^k}{c+a} \geqslant \frac{k}{2}c - \frac{1}{4}(c+a) - \frac{k-2}{2}.$$

Adding the three inequalities above, we obtain

$$\begin{aligned}
&\frac{a^k}{a+b} + \frac{b^k}{b+c} + \frac{c^k}{c+a} \\
\geqslant &\frac{k}{2}(a+b+c) - \frac{1}{2}(a+b+c) - \frac{3}{2}(k-2) \\
= &\frac{k-1}{2}(a+b+c) - \frac{3}{2}(k-2) \\
\geqslant &\frac{3}{2}(k-1) - \frac{3}{2}(k-2) \\
= &\frac{3}{2},
\end{aligned}$$

as desired.

Remark The problem could also be proved by the *Cauchy* inequality or the *Chebyshev* inequality.

2008 (Longyan, Fujian)

The fifth (2008) China Southeastern Mathematical Olympiad was held from 26 to 30 July 2007 in Longyan, Fujian, China, and was hosted by Southeastern Mathematical Society and

Longyan No. 1 High School.

The Competition Committee comprises the following: Li Shenghong, Tao Pingsheng, Zhang Pengcheng, Sun Wenxian, Yang Xiaoming, Zhang Zhengjie, Wu Weichao and Zheng Zhongyi.

First Day

0800 - 1200 July 27, 2008

1. It is given the set $S = \{1, 2, 3, \cdots, 3n\}$, where n is a positive integer. T is a subset of S such that: for any $x, y, z \in T$ (where x, y, z can be the same), $x + y + z \notin T$. Find the maximum value of the number of elements in such set.

Solution Set $T_0 = \{n+1, n+2, \cdots, 3n\}$, where $|T_0| = 2n$ and the sum of any three elements in T_0 is larger than $3n$, that is to say, the sum does not belong to T_0. Thus, $\max|T| \geqslant 2n$.

On the other hand, construct a sequence of sets

$$A_0 = \{n, 2n, 3n\},$$
$$A_k = \{k, 2n-k, 2n+k\}, k = 1, 2, \cdots, n-1,$$

then $S = \bigcup_{k=0}^{n-1} A_k$. For any subset T' in S which has $2n+1$ elements, it must contain a certain A_k.

If $A_0 \subset T'$, it contains the element $3n = n + n + n$.

If a certain $A_k \subset T'$, $k \in \{1, 2, \cdots, n-1\}$, it will contain the element

$$2n + k = k + k + (2n - k),$$

then $\max|T| < 2n+1$. That is, $\max|T| = 2n$.

2 It is given the sequence $\{a_n\}$: $a_1 = 1$,

$$a_{n+1} = 2a_n + n \cdot (1 + 2^n),\ n = 1,\ 2,\ 3,\ \cdots.$$

Find the general term a_n.

Solution Divide the recursion formula by 2^{n+1} throughout, we obtain

$$\frac{a_{n+1}}{2^{n+1}} = \frac{a_n}{2^n} + \frac{n}{2^{n+1}} + \frac{n}{2},$$

that is,

$$\frac{a_{n+1}}{2^{n+1}} - \frac{a_n}{2^n} = \frac{n}{2^{n+1}} + \frac{n}{2}.$$

Then

$$\sum_{i=1}^{n}\left(\frac{a_{i+1}}{2^{i+1}} - \frac{a_i}{2^i}\right) = \sum_{i=1}^{n}\frac{i}{2^{i+1}} + \sum_{i=1}^{n}\frac{i}{2},$$

$$\frac{a_{n+1}}{2^{n+1}} - \frac{a_1}{2^1} = \frac{n(n+1)}{4} + \sum_{i=1}^{n}\frac{i}{2^{i+1}},$$

$$a_{n+1} = 2^{n+1}\left[\frac{n(n+1)}{4} + \frac{1}{2^n} + \frac{1}{2}\sum_{i=1}^{n}\frac{i}{2^i}\right].$$

Set $S_n = \sum_{i=1}^{n}\frac{i}{2^i}$, then $2S_n = \sum_{i=1}^{n}\frac{i}{2^{i-1}}$, and

$$\begin{aligned} S_n &= 2S_n - S_n = \sum_{i=1}^{n}\frac{i}{2^{i-1}} - \sum_{i=1}^{n}\frac{i}{2^i} \\ &= \sum_{i=1}^{n}\frac{i}{2^{i-1}} - \sum_{i=2}^{n+1}\frac{i-1}{2^{i-1}} \\ &= \frac{1}{2^{1-1}} - \frac{n+1-1}{2^{n+1-1}} + \sum_{i=2}^{n}\left(\frac{i}{2^{i-1}} - \frac{i-1}{2^{i-1}}\right) \\ &= 1 - \frac{n}{2^n} + \sum_{i=2}^{n}\frac{1}{2^{i-1}} \end{aligned}$$

$$= 1 - \frac{n}{2^n} + \frac{\frac{1}{2}}{1 - \frac{1}{2}}\left[1 - \left(\frac{1}{2}\right)^{n-1}\right]$$

$$= 1 - \frac{n}{2^n} + 1 - \frac{1}{2^{n-1}}$$

$$= 2 - \frac{n+2}{2^n}.$$

Thus,

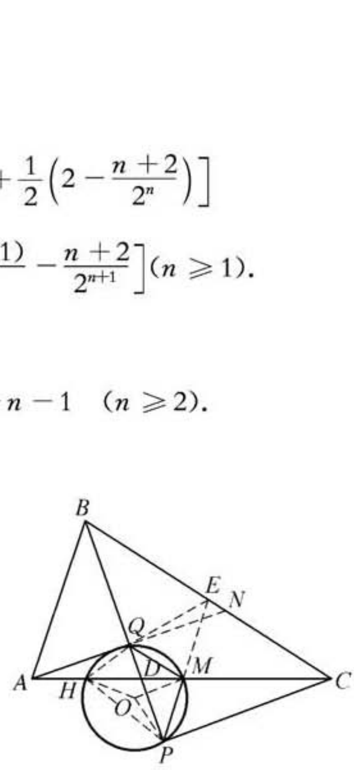
$$a_{n+1} = \left[\frac{n(n+1)}{4} + \frac{1}{2^n} + \frac{1}{2}\left(2 - \frac{n+2}{2^n}\right)\right]$$

$$= 2^{n+1}\left[\frac{3}{2} + \frac{n(n+1)}{4} - \frac{n+2}{2^{n+1}}\right](n \geqslant 1).$$

Consequently,

$$a_n = 2^{n-2}(n^2 - n + 6) - n - 1 \quad (n \geqslant 2).$$

3 In $\triangle ABC$, $BC > AB$, BD bisects $\angle ABC$ and intersects AC at D. As shown in the figure, $CP \perp BD$ with P as the foot of perpendicular and $AQ \perp BP$ with Q as the foot of perpendicular. Points M and E are the midpoints of AC and BC respectively. The circumscribed circle O of $\triangle PQM$ intersects AC at the point H. Prove that O, H, E, M are concyclic.



Proof Extend AQ to intersect BC at point N, then Q, M are the midpoints of AN and AC respectively. Thus $QM \parallel BC$, and

$$\angle PQM = \angle PBC = \frac{1}{2}\angle ABC.$$

Similarly, $\angle MPQ = \frac{1}{2}\angle ABC$. Then $QM = PM$.

Since Q, H, P, M are concyclic, we have

$$\angle PHC = \angle PHM = \angle PQM,$$

that is, $\angle PHC = \angle PBC$. Therefore, P, H, B, C are concyclic and

$$\angle BHC = \angle BPC = 90^\circ.$$

Hence,

$$HE = \frac{1}{2}BC = EP.$$

Since $OH = OP$, we know that OE is the perpendicular bisector of HP. Also, $\angle MPQ = \frac{1}{2}\angle ABC$ and E is the midpoint of BC, we conclude that P, M and E are collinear. Consequently,

$$\angle EHO = \angle EPO = \angle OPM = \angle OMP$$

and O, H, E, M are concyclic.

4 Given positive integers m, $n \geqslant 2$, first select two different a_i, a_j $(j > i)$ in the integer set $A = \{a_1, a_2, \cdots, a_n\}$ and take the difference $a_j - a_i$. Then arrange the $\binom{n}{2}$ differences in ascending order to form a new sequence, which we call 'derived sequence' and is denoted by $\overline{A}$. The number of elements in $\overline{A}$ that can be divided by m is denoted by $\overline{A}(m)$. Prove that for any $m \geqslant 2$, the corresponding derived sequences $\overline{A}$ and $\overline{B}$, with regard to $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{1, 2, \cdots, n\}$, satisfy the inequality $\overline{A}(m) \geqslant \overline{B}(m)$.

Proof For any integer $m \geqslant 2$, if the remainder of x divided by m is i, $i \in \{0, 1, \cdots, m-1\}$, then x belongs to the residue class modulus m, K_i.

Suppose in the set $A = \{a_1, a_2, \cdots, a_n\}$, the number of elements that belong to K_i is $n_i (i = 0, 1, 2, \cdots, m-1)$, while in the set $B = \{1, 2, \cdots, n\}$, the number of elements that belong to K_i is $n_i' (i = 0, 1, 2, \cdots, m-1)$, then

$$\sum_{i=0}^{m-1} n_i = \sum_{i=0}^{m-1} n_i' = n. \qquad ①$$

It is obvious that for every i, j, $|n_i' - n_j'| \leqslant 1$, and $x - y$ is a multiple of m if and only if x, y belong to the same residue class. As to any two elements a_i, a_j in K_i, we have $m \mid a_j - a_i$. Hence, the n_i elements in K_i form $\binom{n_i}{2}$ multiples of m. Considering all the i, we obtain

$$\bar{A}(m) = \sum_{i=0}^{m-1} \binom{n_i}{2}.$$

Similarly,

$$\bar{B}(m) = \sum_{i=0}^{m-1} \binom{n_i'}{2}.$$

Hence, to solve the problem, we just need to prove that $\sum_{i=0}^{m-1} \binom{n_i}{2} \geqslant \sum_{i=0}^{m-1} \binom{n_i'}{2}$, and it can be simplified to

$$\sum_{i=0}^{m-1} n_i^2 \geqslant \sum_{i=0}^{m-1} n_i'^2. \qquad ②$$

From ①, if for every i, j, $|n_i - n_j| \leqslant 1$, then $n_0, n_1, \cdots, n_{m-1}$ and $n_0', n_1', \cdots, n_{m-1}'$ must be the same group (in spite of the different order), and equality holds in ②. Otherwise, if there exist i, j, such that $n_i - n_j \geqslant 2$, then we should just change the two elements n_i, n_j for $\bar{n}_i$, $\bar{n}_j$ respectively, where $\bar{n}_i = n_i - 1$, $\bar{n}_j = n_j + 1$, and $n_i + n_j = \bar{n}_i + \bar{n}_j$. Since

$$(n_i^2 + n_j^2) - (\bar{n}_i^2 + \bar{n}_j^2) = 2(n_i - n_j - 1) > 0,$$

the sum of the left side in ② will decrease after adjustment. Therefore, the minimum value of ② is attained if and only if n_0, n_1, $\cdots$, n_{m-1} and n'_0, n'_1, $\cdots$, n'_{m-1} are the same group (in spite of the different order), that is, the inequality ② holds.

Second Day

0800 - 1230 July 28, 2008

5 Find the largest positive number λ such that

$$|\lambda xy + yz| \leqslant \frac{\sqrt{5}}{2}, \text{ where } x^2 + y^2 + z^2 = 1.$$

Solution Note that

$$\begin{aligned} 1 &= x^2 + y^2 + z^2 \\ &= x^2 + \frac{\lambda^2}{1+\lambda^2} y^2 + \frac{1}{1+\lambda^2} y^2 + z^2 \\ &\geqslant \frac{2}{\sqrt{1+\lambda^2}} (\lambda |xy| + |yz|) \\ &\geqslant \frac{2}{\sqrt{1+\lambda^2}} (|\lambda xy + yz|), \end{aligned}$$

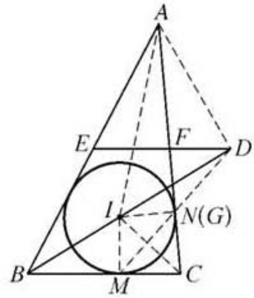
and the two equalities hold simultaneously when

$$y = \frac{\sqrt{2}}{2},\ x = \frac{\sqrt{2}\lambda}{2\sqrt{\lambda^2+1}},\ z = \frac{\sqrt{2}}{2\sqrt{\lambda^2+1}}.$$

Thus, $\frac{\sqrt{1+\lambda^2}}{2}$ is the maximum value of $|\lambda xy + yz|$. Let

$\frac{\sqrt{1+\lambda^2}}{2}=\frac{\sqrt{5}}{2}$. We obtain that $\lambda=2$.

6 As shown in the figure, BC and AC are tangent to the inscribed circle I of $\triangle ABC$ at M and N. E and F are the midpoints of AB, AC respectively. EF intersects BI at D. Prove that M, N, D are collinear.



Proof Join AD, and it is obvious that $\angle ADB=90°$. Then join AI and DM. Suppose DM intersects AC at G. Since $\angle ABI=\angle DBM$, we obtain $\frac{AB}{BD}=\frac{BI}{BM}$. Hence, $\triangle ABI \backsim \triangle DBM$, and

$$\angle DMB=\angle AIB=90°+\frac{1}{2}\angle ACB.$$

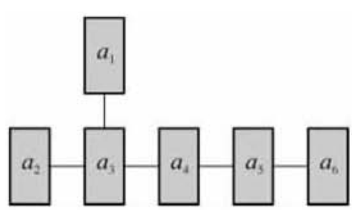
Join IG, IC, IM, then

$$\angle IMG=\angle DMB-90°=\frac{1}{2}\angle ACB=\angle GCI.$$

Therefore, I, M, C, G are concyclic, and $IG\perp AC$.

Consequently, since G and N represent the same point, we conclude that M, N, D are collinear.

7 Captain Jack and his pirates robbed 6 boxes of gold coins A_1, A_2, A_3, A_4, A_5, A_6. There are a_i coins in box A_i ($i=1$, 2,

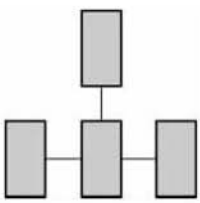


3, 4, 5, 6) and $a_i \neq a_j (i \neq j)$. They laid the boxes as shown in the figure. Captain Jack would take turns with a nominated pirate to choose a box. The rule was: Each person could only choose a box which was adjacent to at most one box. If Captain Jack got more gold coins than the pirates, then Captain won the game. If Captain Jack went first, what should be his strategy to win the game?

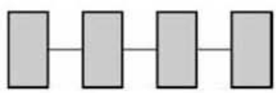
Solution When there are 2 boxes, Captain Jack will naturally win the game.

Lemma 1 *When there are* 4 *boxes, Captain Jack would also have a strategy to win the game.*

How is it? Since there are 4 boxes, then there are two ways to link them.



Case 1



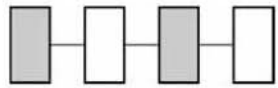
Case 2

Case 1

In the first round, Captain Jack has the three outer boxes to choose from, it is certain he will choose the one having most coins, while the pirate could only choose one from the other two. Anyway, they cannot choose the one in the centre. After the first round, Captain Jack would have more coins than the pirate. Then, there are only 2 remaining boxes, Captain Jack will surely choose the one having more coins and win the game.

Case 2

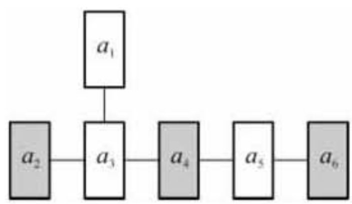
Paint 4 boxes in black and white and arrange as shown in the figure. If there are more coins in black boxes than in the white ones, Captain Jack will take the black box first, forcing pirate to take the white one, and after that, Captain Jack will take the other black box and win the game.



Now let us examine the original problem.

If $a_6 \geqslant a_5$, Captain Jack could take the available box that has the most coins, then the pirate takes one. And this problem would be simplified to "4 boxes" problem in Lemma 1.

Otherwise, if $a_5 > a_6$, we can assume $a_1 > a_2$, and paint a_1, a_3, a_5 black, while the rest white, as shown in the figure. Then, we should check whether if $a_1 + a_3 + a_5 \geqslant a_2 + a_4 + a_6$ or not. If it is, Captain Jack could take all the white boxes and win the game. If not, he could take a_6 first, then



(1) If the pirate takes a_1, then Captain Jack takes a_2 and a_4 to win the game.

(2) If the pirate takes a_2, since $a_1 > a_2$, then Captain Jack takes a_1. Although he could not have taken all the black boxes, but since $a_1 > a_2$, he could also win the game.

(3) If the pirate takes a_5, then Captain Jack takes a_4, and

(a) If the pirate takes a_1, Captain Jack could take a_2 to win

(b) If the pirate takes a_2, since $a_1 > a_2$, Captain Jack could take a_1 to win.

Therefore, Captain Jack has always a strategy to win the game.

8 Let n be a positive integer, and $f(n)$ denote the number of n-digit integers $\overline{a_1a_2\cdots a_n}$ (called wave number) that satisfy the following conditions:

(i) $a_i \in \{1, 2, 3, 4\}$, and $a_i \neq a_{i+1}$, $i = 1, 2, \cdots$;

(ii) When $n \geqslant 3$, the numbers $a_i - a_{i+1}$ and $a_{i+1} - a_{i+2}$ have opposite signs, $i = 1, 2, \cdots$.

Find (1) the value of $f(10)$,

(2) the remainder of $f(2008)$ divided by 13.

Solution (1) When $n \geqslant 2$, if $a_1 < a_2$, $\overline{a_1a_2\cdots a_n}$ is classified as A class. The number of $\overline{a_1a_2\cdots a_n}$ is denoted by $g(n)$. If $a_1 > a_2$, then $\overline{a_1a_2\cdots a_n}$ is classified as B class. By symmetry, the number of such $\overline{a_1a_2\cdots a_n}$ is also $g(n)$. Thus, $f(n) = 2g(n)$.

Now we want to find $g(n)$. Denote $m_k(i)$ as the k-digit "A wave number" whose last digit is i ($i = 1, 2, 3, 4$), then

$$g(n) = \sum_{i=1}^{4} m_n(i).$$

As $a_{2k-1} < a_{2k}$, $a_{2k} > a_{2k+1}$, we have the following 2 cases.

(a) When k is even, $m_{k+1}(4) = 0$, $m_{k+1}(3) = m_k(4)$, $m_{k+1}(2) = m_k(4) + m_k(3)$, $m_{k+1}(1) = m_k(4) + m_k(3) + m_k(2)$.

(b) When k is odd, $m_{k+1}(1) = 0$, $m_{k+1}(2) = m_k(1)$, $m_{k+1}(3) = m_k(1) + m_k(2)$, $m_{k+1}(4) = m_k(1) + m_k(2) + m_k(3)$. It is obvious that $m_2(1) = 0$, $m_2(2) = 1$, $m_2(3) = 2$, $m_2(4) = 3$, then, $g(2) = 6$.

Hence,

$$m_3(1) = m_2(2) + m_2(3) + m_2(4) = 6,$$

$$m_3(2) = m_2(3) + m_2(4) = 5,$$

$$m_3(3) = m_2(4) = 3,\ m_3(4) = 0.$$

Therefore

$$g(3)=\sum_{i=1}^{4}m_3(i)=14.$$

On the other hand, since

$$m_4(1)=0,\ m_4(2)=m_3(1)=6,$$

$$m_4(3)=m_3(1)+m_3(2)=11,$$

$$m_4(4)=m_3(1)+m_3(2)+m_3(3)=14,$$

we obtain,

$$g(4)=\sum_{i=1}^{4}m_4(i)=31.$$

In the same way, we could get $g(5)=70$, $g(6)=157$, $g(7)=353$, $g(8)=793$.

Then, in general, when $n\geqslant 5$,

$$g(n)=2g(n-1)+g(n-2)-g(n-3). \qquad ③$$

Now we prove ③ as follows.

Using mathematical induction, we are done when $n=5$, 6, 7, 8. Suppose ③ holds when for 5, 6, 7, 8⋯, n, now consider the case for $n+1$. When n is even, from (a), (b), we have

$$m_{n+1}(4)=0,\ m_{n+1}(3)=m_n(4),$$

$$m_{n+1}(2)=m_n(4)+m_n(3),$$

$$m_{n+1}(1)=m_n(4)+m_n(3)+m_n(2).$$

As $m_n(1)=0$, then

$$\begin{aligned}g(n+1)&=\sum_{i=1}^{4}m_{n+1}(i)\\&=2\Big(\sum_{i=1}^{4}m_n(i)\Big)+m_n(4)-m_n(2)\end{aligned}$$

$$= 2g(n) + m_n(4) - m_n(2).$$

Since

$$\begin{aligned} m_n(4) &= m_{n-1}(1) + m_{n-1}(2) + m_{n-1}(3) + 0 \\ &= \sum_{i=1}^{4} m_{n-1}(i) = g(n-1), \\ m_n(2) &= m_{n-1}(1) = m_{n-2}(4) + m_{n-2}(3) + m_{n-2}(2) + 0 \\ &= g(n-2). \end{aligned}$$

We obtain,

$$g(n+1) = 2g(n) + g(n-1) - g(n-2).$$

On the other hand, when n is odd, $g(n+1) = \sum_{i=1}^{4} m_{n+1}(i)$. Since $m_{n+1}(1) = 0$, $m_{n+1}(2) = m_n(1)$, $m_n(4) = 0$,

$$m_{n+1}(3) = m_n(1) + m_n(2),$$

$$m_{n+1}(4) = m_n(1) + m_n(2) + m_n(3),$$

then,

$$\begin{aligned} g(n+1) &= \sum_{i=1}^{4} m_{n+1}(i) \\ &= 2\sum_{i=1}^{4} m_n(i) + m_n(1) - m_n(3) \\ &= 2g(n) + m_n(1) - m_n(3). \end{aligned}$$

Since

$$\begin{aligned} m_n(1) &= m_{n-1}(4) + m_{n-1}(3) + m_{n-1}(2) + 0 = g(n-1), \\ m_n(3) &= m_{n-1}(4) = m_{n-2}(1) + m_{n-2}(2) + m_{n-2}(3) + 0 \\ &= g(n-2). \end{aligned}$$

We get

$$g(n+1) = 2g(n) + g(n-1) - g(n-2).$$

Hence, ③ holds for $n+1$. By mathematical induction, ③ holds when $n \geqslant 5$.

From ③,

$$g(9) = 2g(8) + g(7) - g(6) = 1782,$$
$$g(10) = 2g(9) + g(8) - g(7) = 4004.$$

Thus,

$$f(10) = 2g(10) = 8008.$$

(2) Now consider the sequence of remainders of $\{g(n)\}$ divided by 13. From ③, when $n = 2, 3, 4, \cdots, 14, 15, 16, 17, \cdots$, the corresponding remainders are $6, 1, 5, 5, 1, 2, 0, 1, 0, 1, 1, 3; 6, 1, 5, 5, \cdots$

Therefore, when $n \geqslant 2$, the sequence of remainders is a periodic sequence whose minimum period is 12. As

$$2008 = 12 \times 167 + 4,$$

we get

$$g(2008) \equiv 5 \pmod{13}.$$

Therefore,

$$f(2008) \equiv 10 \pmod{13}.$$

International Mathematical Olympiad

2007 (Hanoi, Vietnam)

First Day

0900 – 1300 July 25, 2007

1. Real numbers $a_1, a_2, \cdots, a_n$ are given. For each i $(1 \leqslant i \leqslant n)$, define

$$d_i = \max\{a_j : 1 \leqslant j \leqslant i\} - \min\{a_j : i \leqslant j \leqslant n\},$$

and let

$$d = \max\{d_i : 1 \leqslant i \leqslant n\}.$$

(1) Prove that: for any real numbers $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$

$$\max\{|x_i - a_i| : 1 \leqslant i \leqslant n\} \geqslant \frac{d}{2}. \qquad ①$$

(2) Show that there are real numbers $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ such that equality holds in ①.

Solution (1) Define

$$d = d_g (1 \leqslant g \leqslant n),$$
$$a_p = \max\{a_j : 1 \leqslant j \leqslant g\},$$
$$a_r = \min\{a_j : g \leqslant j \leqslant n\}.$$

This yields $1 \leqslant p \leqslant g \leqslant r \leqslant n$ and $d = a_p - a_r$.

It is worthwhile to observe that for any real numbers $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$,

$$\begin{aligned}(a_p - x_p) + (x_r - a_r) &= (a_p - a_r) + (x_r - x_p) \\ &\geqslant a_p - a_r = d.\end{aligned}$$

Consequently,

$$a_p - x_p \geqslant \frac{d}{2} \text{ or } x_r - a_r \geqslant \frac{d}{2}.$$

Hence

$$\begin{aligned}\max\{|x_i - a_i| : 1 \leqslant i \leqslant n\} &\geqslant \max\{|x_p - a_p|, |x_r - a_r|\} \\ &\geqslant \max\{a_p - x_p, x_r - a_r\} \\ &\geqslant \frac{d}{2}.\end{aligned}$$

(2) Define the sequence $\{x_k\}$ as $x_1 = a_1 - \frac{d}{2}$, $x_k = \max\left\{x_{k-1}, a_k - \frac{d}{2}\right\}$, $(2 \leqslant k \leqslant n)$.

Now we will prove that for the above sequence the equality holds in ①.

By definition of $\{x_k\}$, $\{x_k\}$ is a non-decreasing sequence, and $x_k - a_k \geqslant -\frac{d}{2}$ for all k $(1 \leqslant k \leqslant n)$.

In the following, we prove for all k $(1 \leqslant k \leqslant n)$

$$x_k - a_k \leqslant \frac{d}{2}. \qquad ②$$

For any k $(1 \leqslant k \leqslant n)$, let l $(l \leqslant k)$ be the smallest integer such that $x_k = x_l$. Hence $l = 1$ or $l \geqslant 2$ and $x_l > x_{l-1}$.

In both cases,

$$x_k = x_l = a_l - \frac{d}{2} \qquad ③$$

holds.

Since

$$a_l - a_k \leqslant \max\{a_j : 1 \leqslant j \leqslant k\} - \min\{a_j : k \leqslant j \leqslant n\} \leqslant d,$$

in view of ③, we know

$$x_k - a_k = a_l - a_k - \frac{d}{2} \leqslant d - \frac{d}{2} = \frac{d}{2}.$$

This establishes ②. Thus

$$-\frac{d}{2} \leqslant x_k - a_k \leqslant \frac{d}{2}$$

holds for all $1 \leqslant k \leqslant n$, hence also

$$\max\{\mid x_i - a_i \mid : 1 \leqslant i \leqslant n\} \leqslant \frac{d}{2}.$$

In view of part (1), the equality holds in ① for the sequence $\{x_k\}$.

An alternative solution of part (2).

For each i $(1 \leqslant i \leqslant n)$, define

$$M_i = \max\{a_j : 1 \leqslant j \leqslant i\}, \quad m_i = \min\{a_j : i \leqslant j \leqslant n\}.$$

Then

$$M_i = \max\{a_1, \cdots, a_i\} \leqslant \max\{a_1, \cdots, a_i, a_{i+1}\} = M_{i+1},$$
$$m_i = \min\{a_i, a_{i+1}, \cdots, a_n\} \leqslant \min\{a_{i+1}, \cdots, a_n\} = m_{i+1}.$$

It is also worthwhile to observe that $m_i \leqslant a_i \leqslant M_i$.

Take $x_i = \dfrac{M_i + m_i}{2}$, by $d_i = M_i - m_i$ yields

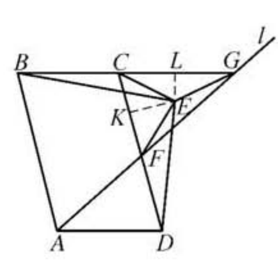
$$-\frac{d}{2} = \frac{m_i - M_i}{2} = x_i - M_i \leqslant x_i - a_i$$
$$\leqslant x_i - m_i = \frac{M_i - m_i}{2} = \frac{d_i}{2}.$$

Consequently,

$$\max\{|x_k - a_k| : 1 \leqslant k \leqslant n\} \leqslant \max\left\{\frac{d_i}{2} : 1 \leqslant i \leqslant n\right\} = \frac{d}{2}.$$

In view of part (1), the equality holds in ① for the sequence $\{x_k\}$.

2 Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let l be a line passing through A. Suppose that l intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that l is the bisector of $\angle DAB$.



Proof Draw the altitudes of two isosceles triangles EGC and ECF as in the figure.

In view of the given condition, it is easy to see that $\triangle ADF \backsim \triangle GCF$. Hence

$$\frac{AD}{GC}=\frac{DF}{CF}\Rightarrow\frac{BC}{CG}=\frac{DF}{CF}\Rightarrow\frac{BC}{CL}=\frac{DF}{CK}$$

$$\Rightarrow\frac{BC+CL}{CL}=\frac{DF+FK}{CK}$$

$$\Rightarrow\frac{BL}{CL}=\frac{DK}{CK}$$

$$\Rightarrow\frac{BL}{DK}=\frac{CL}{CK}. \quad ①$$

Since $BCED$ is a cyclic quadrilateral, $\angle LBE=\angle EDK$, this yields $\triangle BLE \backsim \triangle DKE$, where both are right-angled triangles.

So

$$\frac{BL}{DK}=\frac{EL}{EK}. \quad ②$$

In view of ① and ②, $\frac{CL}{CK}=\frac{EL}{EK}$, this means $\triangle CLE \backsim \triangle CKE$. Thus

$$\frac{CL}{CK}=\frac{CE}{CE}=1,$$

i.e. $CL=CK\Rightarrow CG=CF$.

It is intuitively obvious that $\angle BAG=\angle GAD$. Hence l is the bisector.

3 In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size.

Given that, in this competition, the size of the largest clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Proof We provide an algorithm to distribute the competitors.

Denote the rooms A and B. At some initial stage, we move one person at a time from one room to the other one. We achieve the target by going through several adjustments. In every step of the algorithm, let A and B be the sets of competitors in room A and B. Let $C(A)$ and $C(B)$ be the largest size of clique in room A and B respectively.

Step 1: Let M be the largest clique of all competitors, $|M| = 2m$.

Move all members of M to room A, and the remained ones to room B.

Since M is the largest clique of all competitors, we have $C(A) = |M| \geqslant C(B)$.

Step 2: If $C(A) > C(B)$, move one person from room A to room B. (In view of $C(A) > C(B)$, we have $A \neq \varnothing$.)

After every operation is done, $C(A)$ decreases by 1 while $C(B)$ increases by 1 at most. These operations will not end until

$$C(A) \leqslant C(B) \leqslant C(A) + 1.$$

At that time, we also have $C(A) = |A| \geqslant m$. (Otherwise there are at least $m+1$ members of M in room B, at most $m-1$ members in room A, then $C(B) - C(A) \geqslant (m+1) - (m-1) = 2$, it is impossible.)

Step 3: Denote $K = C(A)$. If $C(B) = K$, we are done; or else $C(B) = K+1$. From the above discussion,

$$K = |A| = |A \cap M| \geqslant m, \ |B \cap M| \leqslant m.$$

Step 4: If there is a clique C in room B with $|C|=K+1$ and a competitor $x \in B \cap M$ but $x \notin C$, then move x to room A, we are done.

In fact, after the operation, there are $K+1$ members of M in room A, so $C(A)=K+1$. Since $x \notin C$, whose removal does not reduce C, $C(B)=C$. Therefore, $C(A)=C(B)=K+1$.

If such competitor x does not exist, then each largest clique in room B contains $B \cap M$ as a subset. In this case, we do step 5.

Step 5: Choose any of the largest clique C ($|C|=K+1$) in room B, move a member of $C \setminus M$ to room A. (In view of $|C|=K+1>m \geqslant |B \cap M|$, we know $C \setminus M \neq \varnothing$.)

Since we only move one person at a time from room B to room A, so $C(B)$ decreases by 1 at most. At the end of this step, we have $C(B)=K$.

Now, there is a clique $A \cap M$ in A, $|A \cap M|=K$. So $C(A) \geqslant K$.

We prove $C(A)=K$ as follows.

Let Q be any clique of room A. We only need to show $|Q| \leqslant K$.

In fact, the members of room A can be classified into two:

(1) Some members of M in view of M being a clique, are friends with all the members of $B \cap M$.

(2) The members move from room B to room A at step 5, they are friends of $B \cap M$.

So, every member of Q and members of $B \cap M$ are friends. What is more, Q and $B \cap M$ both are cliques, so is $Q \cup (B \cap M)$.

Since M is the largest clique of all competitors,

$$|M| \geqslant |Q \cup (B \cap M)| = |Q| + |B \cap M|$$

$$= | Q | + | M | - | A \cap M |.$$

So $| Q | \leqslant | A \cap M | = K$. Therefore, after these 5 steps, we have $C(A) = C(B) = K$.

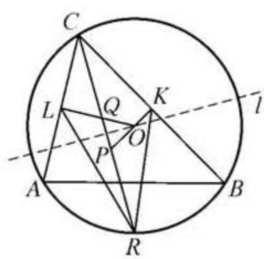
Second Day

0900 - 1330 July 26, 2007

4 In triangle ABC the bisector of angle BCA intersects the circumcircle at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Proof

If $AC = BC$, $\triangle ABC$ is an isosceles triangle, and CR is the symmetry axis of $\triangle RQL$ and $\triangle RPK$. The conclusion is obviously true.



If $AC \neq BC$, without loss of generality, let $AC < BC$. Denote the center of circumcircle of $\triangle ABC$ by O.

Since the right triangles CQL and CPK are similar,

$$\angle CPK = \angle CQL = \angle OQP, \text{ and } \frac{QL}{PK} = \frac{CQ}{CP}. \qquad ①$$

Let l be the perpendicular bisector of CR, then O is on l.

Since $\triangle OPQ$ is an isosceles triangle, P and Q are two points symmetrical about l on CR.

So

$$RP = CQ \quad \text{and} \quad RQ = CP. \qquad ②$$

By ①, ②,

$$\frac{S(\triangle RQL)}{S(\triangle RPK)} = \frac{\frac{1}{2} \cdot RQ \cdot QL \cdot \sin \angle RQL}{\frac{1}{2} \cdot RP \cdot PK \cdot \sin \angle RPK}$$

$$= \frac{RQ}{RP} \cdot \frac{QL}{PK} = \frac{CP}{CQ} \cdot \frac{CQ}{CP} = 1.$$

Hence the two triangles have the same area.

5 Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

Proof Call (a, b) a "bad pair" if it satisfies $4ab - 1 \mid (4a^2 - 1)^2$ while $a \neq b$. We use the method of infinite descent to prove there is no such "bad pair".

Property 1 *If (a, b) is a "bad pair" and $a < b$, there exists an integer c $(c < a)$ such that (a, c) is also a "bad pair".*

In fact, let $r = \frac{(4a^2 - 1)^2}{4ab - 1}$, then

$$r = -r \cdot (-1) = -(4a^2 - 1)^2 \equiv -1 \pmod{4a}.$$

Therefore there exists an integer c such that $r = 4ac - 1$. Since $a < b$, we have

$$4ac - 1 = \frac{(4a^2 - 1)^2}{4ab - 1} < 4a^2 - 1.$$

So $c < a$ and $4ac - 1 \mid (4a^2 - 1)^2$. Thus (a, c) is a "bad pair" too.

Property 2 *If (a, b) is a "bad pair", so is (b, a).*

In fact, by

$$1 = 1^2 \equiv (4ab)^2 (\text{mod } (4ab - 1)),$$

we get

$$(4b^2 - 1)^2 \equiv (4b^2 - (4ab)^2)^2 = 16b^4(4a^2 - 1)^2$$
$$\equiv 0 (\bmod (4ab - 1)).$$

Thus $4ab - 1 \mid (4b^2 - 1)^2$.

In the following we will show that such "bad pair" does not exist. We shall prove by contradiction.

Suppose there is at least one "bad pair", we choose such pair for which $2a + b$ is minimum.

If $a < b$, by property 1, there is a "bad pair" (a, c) which satisfies $c < b$, and $2a + c < 2a + b$, a contradiction.

If $b < a$, by property 2, (b, a) is also a "bad pair", which leads to $2b + a < 2a + b$, a contradiction.

Hence such "bad pair" does not exist. Therefore $a = b$.

6 Let n be a positive integer. Consider

$$S = \{(x, y, z): x, y, z \in \{0, 1, \cdots, n\},$$
$$x + y + z > 0\},$$

a set of $(n + 1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

Solution The answer is: we need at least $3n$ planes.

It is easy to see that $3n$ planes satisfy the given conditions. For example, $x = i$, $y = i$ and $z = i$ $(i = 1, 2, \cdots, n)$. Another example is $x + y + z = k$ $(k = 1, 2, \cdots, 3n)$.

Now we prove that $3n$ is the extreme value. The following lemma is the key to the proof.

Lemma 1 *Let $P(x_1, \cdots, x_k)$ be a non-zero polynomial in n variables. If every n-tuple $(x_1, \cdots, x_k)$, satisfying $x_1, \cdots, x_k \in \{0, 1, \cdots, n\}$ as well as $x_1 + x_2 + \cdots + x_k > 0$, is the zero point of the polynomial, and $P(0, 0, \cdots, 0) \neq 0$, then*

$\deg P \geqslant kn$.

Now we prove Lemma 1.

We apply induction on k.

For $k = 0$, since $P \neq 0$, the lemma is surely true.

Assuming that the lemma holds for $k - 1$, we prove that it also holds for k.

Denote $y = x_k$, let $R(x_1, \cdots, x_{k-1}, y)$ be the remainder when P is divided by $Q(y) = y(y-1)(y-2)\cdots(y-n)$.

Since $Q(y) = y(y-1)(y-2)\cdots(y-n)$ has $n+1$ zeroes, namely $y = 0, 1, \cdots, n$. This yields

$$P(x_1, \cdots, x_{k-1}, y) = R(x_1, \cdots, x_{k-1}, y)$$

for all $x_1, \cdots, x_k, y \in \{0, 1, \cdots, n\}$. This means R satisfies the hypothesis of the lemma, what is more, $\deg_y R \leqslant n$. Also it is obvious that $\deg R \leqslant \deg P$. So we only need to show that $\deg R \geqslant nk$.

Now, rewrite R in descending powers of y:

$$\begin{aligned}&R(x_1, \cdots, x_{k-1}, y)\\ =\ &R_n(x_1, \cdots, x_{k-1})y^n + R_{n-1}(x_1, \cdots, x_{k-1})y^{n-1} + \cdots\\ &+ R_0(x_1, \cdots, x_{k-1}).\end{aligned}$$

In what follows, we prove that $R_n(x_1, \cdots, x_{k-1})$ satisfies the induction hypothesis.

In fact, consider the polynomial $T(y) = R(0, \cdots, 0, y)$. It is easy to see that $\deg T(y) \leqslant n$, and $T(y)$ has n roots: $y = 1, \cdots, n$.

On the other hand, since $T(0) \neq 0$ then $T(y) \neq 0$. So $\deg T(y) = n$ and $R_n(0, \cdots, 0) \neq 0$. (In the case of $k = 1$, the coefficient R_n is non-zero.)

For each fixed $a_1, \cdots, a_{k-1} \in \{0, 1, \cdots, n\}$ and $a_1 + \cdots +$

$a_{k-1} > 0$, the polynomial $R(a_1, \cdots, a_{k-1}, y)$ obtained from $R(x_1, \cdots, x_{k-1}, y)$ by substituting the value of $x_i = a_i$ has $n+1$ roots while $\deg R \leqslant n$. Hence it is a zero polynomial, and $R_i(a_1, \cdots, a_{k-1}) = 0$ $(i = 0, 1, \cdots, n)$. In particular, $R_n(a_1, \cdots, a_{k-1}) = 0$.

By the induction hypothesis, $\deg R_n \geqslant (k-1)n$. So

$$\deg R \geqslant \deg R_n + n \geqslant kn.$$

This completes the induction and the proof of Lemma 1.

Assume the union of N planes contains S but does not include $(0, 0, 0)$, let these N planes be

$$a_i x + b_i y + c_i z + d_i = 0.$$

Consider polynomial

$$P(x, y, z) = \prod_{i=1}^{N} (a_i x + b_i y + c_i z + d_i).$$

The degree of it is N.

For any $(x_0, y_0, z_0) \in S$, $P(x_0, y_0, z_0) = 0$, but $P(0, 0, 0) \neq 0$. By lemma, we get $N = \deg P \geqslant 3n$.

Remark This problem belongs to real algebraic geometry, it is noteworthy.

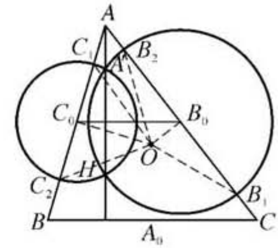
2008 (Madrid, Spain)

First Day

0900 – 1330 July 16, 2008

1 An acute triangle ABC has orthocenter H. The circle

through H with center the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with center the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with center the midpoint of AB intersects the line AB at C_1 and C_2.



Show that A_1, A_2, B_1, B_2, C_1, C_2 are concyclic.

Proof I Let B_0, C_0 the midpoints of CA, ABrespectively. Denote A' as the other intersection of the circle centered at B_0 which passes through H and the circle centered at C_0 which passes through H. We know that $A'H \perp C_0B_0$. Since B_0, C_0 are the midpoints of CA, AB respectively, $B_0C_0 \parallel BC$. Therefore $A'H \perp BC$. This yields that A' lies on the segment AH.

By the Secant - Secant theorem, it follows that

$$AC_1 \cdot AC_2 = AA' \cdot AH = AB_1 \cdot AB_2,$$

So B_1, B_2, C_1, C_2 are concyclic.

Let the intersection of the perpendicular bisectors of B_1B_2, C_1C_2 be O. Then O is the circumcenter of quadrilateral $B_1B_2C_1C_2$, as well as the circumcenter of $\triangle ABC$. So

$$OB_1 = OB_2 = OC_1 = OC_2.$$

Similarly,

$$OA_1 = OA_2 = OB_1 = OB_2.$$

Therefore, six points A_1, A_2, B_1, B_2, C_1, C_2 are all on the same circle, whose center is O, and radius OA_1.

Proof II Let O be the circumcenter of triangle ABC, and D,

E, F the midpoints of BC, CA, AB respectively. BH intersects DF at point P, then $BH \perp DF$. By *Pythagoras*' theorem, we have

$$BF^2 - FH^2 = BP^2 - PH^2 = BD^2 - DH^2, \quad ①$$

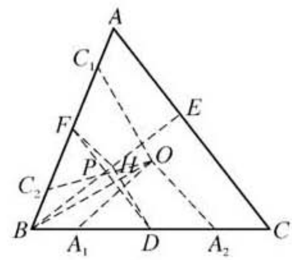
$$BO^2 - A_1O^2 = BD^2 - A_1D^2 = BD^2 - DH^2. \quad ②$$

Similarly,

$$BO^2 - C_2O^2 = BF^2 - FH^2, \quad ③$$

By ①, ②, ③, $A_1O = C_2O$. It is obvious that $A_1O = A_2O$, $C_1O = C_2O$, thus

$$A_1O = A_2O = C_1O = C_2O.$$



Similarly,

$$A_1O = A_2O = B_1O = B_2O.$$

Therefore, six points A_1, A_2, B_1, B_2, C_1, C_2 are all on the same circle, whose center is O.

2 (1) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geqslant 1,$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

(2) Prove that the equality holds for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Proof Ⅰ (1) Let

$$\frac{x}{x-1} = a, \frac{y}{y-1} = b, \frac{z}{z-1} = c,$$

then

$$x = \frac{a}{a-1},\ y = \frac{b}{b-1},\ z = \frac{c}{c-1}.$$

Since $xyz = 1$, we have

$$abc = (a-1)(b-1)(c-1),$$

that is

$$a + b + c - 1 = ab + bc + ca.$$

Therefore

$$\begin{aligned} a^2 + b^2 + c^2 &= (a+b+c)^2 - 2(ab+bc+ca) \\ &= (a+b+c)^2 - 2(a+b+c-1) \\ &= (a+b+c-1)^2 + 1 \\ &\geqslant 1. \end{aligned}$$

So

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geqslant 1.$$

(2) Take $(x, y, z) = \left(-\frac{k}{(k-1)^2}, k - k^2, \frac{k-1}{k^2}\right)$, k is an integer, then (x, y, z) is a triple of rational numbers, with x, y, z each different from 1. What is more, a different integer k gives a different triple of rational numbers.

$$\begin{aligned} &\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \\ = &\frac{k^2}{(k^2-k+1)^2} + \frac{(k-k^2)^2}{(k^2-k+1)} + \frac{(k-1)^2}{(k^2-k+1)} \\ = &\frac{k^4 - 2k^3 + 3k^2 - 2k + 1}{(k^2-k+1)^2} = 1. \end{aligned}$$

So the problem is proved.

Proof Ⅱ (1) By $xyz = 1$, let $p = x$, $q = 1$, $r = \frac{1}{y}$, then $x = \frac{p}{q}$, $y = \frac{q}{r}$, $z = \frac{1}{xy} = \frac{r}{p}$, where p, q, r are different from each other. We have

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geqslant 1$$

$$\Leftrightarrow \frac{p^2}{(p-q)^2} + \frac{q^2}{(q-r)^2} + \frac{r^2}{(r-p)^2} \geqslant 1. \qquad ①$$

Let

$$a = \frac{p}{p-q},\ b = \frac{q}{q-r},\ c = \frac{r}{r-p},$$

after the substitution, ① reduces to $\sum a^2 \geqslant 1$. Since

$$\frac{-1+a}{a} = \frac{q}{p},\ \frac{-1+b}{b} = \frac{r}{q},\ \frac{-1+c}{c} = \frac{p}{r},$$

therefore

$$\frac{-1+a}{a} \cdot \frac{-1+b}{b} \cdot \frac{-1+c}{c} = 1,$$

$$1 - \sum a + \sum ab = 0. \qquad ②$$

By ②, we get

$$1 - \sum a^2 = -(a+b+c-1)^2 \leqslant 0,$$

so $\sum a^2 \geqslant 1$. Hence ① holds.

(2) Let $b = \frac{t^2+t}{t^2+t+1}$, $c = \frac{t+1}{t^2+t+1}$, $a = -\frac{bc}{b+c}$, here t can be any rational number except 0 and -1. While t varies, only finitely many of t values can make b, c, a be 1. That is, there are infinitely many triples of rational numbers a, b, c

each different from 1, satisfying $\sum a = \sum a^2 = 1$. By

$$(x, y, z) = \left(\frac{a}{a-1}, \frac{b}{b-1}, \frac{c}{c-1}\right),$$

(2) holds.

3 Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

Proof I Take any integer m $(m \geqslant 20)$, p is a prime divisor of $(m!)^2 + 1$, then $p > m \geqslant 20$. Take a integer n such that $0 < n < \frac{p}{2}$ and $n \equiv \pm m!$ (mod p). Therefore $0 < n < p - n < p$ and

$$n^2 \equiv -1 \pmod{p}. \quad ①$$

Now

$$(p - 2n)^2 = p^2 - 4pn + 4n^2 \equiv -4 \pmod{p},$$

which yields $(p - 2n)^2 \geqslant p - 4$,

$$\begin{aligned} p &\geqslant 2n + \sqrt{p-4} \\ &\geqslant 2n + \sqrt{2n + \sqrt{p-4} - 4} \\ &> 2n + \sqrt{2n}. \end{aligned} \quad ②$$

By virtue of ① and ②, we are done.

Proof II First, if a prime $p \equiv 1$ (mod 4), then $\left(\frac{-1}{p}\right) = 1$, i.e. there exists an integer $n \in \{1, 2, \cdots, p-1\}$ such that $n^2 \equiv -1$ (mod p). Obviously, for this n,

$$(p - n)^2 \equiv n^2 \equiv -1 \pmod{p}$$

and $\min\{n, p - n\} \leqslant \frac{p-1}{2}$. Therefore, there exists an integer

$$f(p) \in \left\{1, 2, \cdots, \frac{p-1}{2}\right\}$$

such that $f^2(p) \equiv -1 \pmod{p}$.

Next, we will show that if p is sufficiently large ($p \geqslant 29$), $n = f(p)$ satisfies

$$2n + \sqrt{2n} < p. \qquad ③$$

Suppose for some $p \geqslant 29$, the above assumption is false. Then

$$\begin{aligned} 2n + \sqrt{2n} \geqslant p &\Leftrightarrow (1 + 2\sqrt{2n})^2 \geqslant 4p + 1 \\ &\Leftrightarrow n \geqslant \frac{p-1}{2} - \frac{\sqrt{4p+1} - 3}{4}. \end{aligned}$$

Let $t = \frac{p-1}{2} - n \in \mathbf{Z}$, then $0 \leqslant t \leqslant \frac{\sqrt{4p+1} - 3}{4}$. Furthermore,

$$\begin{aligned} 0 \equiv n^2 + 1 &\equiv \left(\frac{p-1}{2} - t\right)^2 + 1 \\ &= \frac{p^2 - 2p + 1}{4} + t^2 - (p-1)t + 1 \\ &\equiv \frac{3p+5}{4} + t^2 + t \pmod{p}, \end{aligned}$$

because $4 \mid p - 5$, so

$$\frac{p^2 - 2p + 1}{4} - \frac{3p+1}{4} = p \cdot \frac{p-5}{4} \equiv 0 \pmod{p}.$$

Therefore

$$p \,\Big|\, t^2 + t + \frac{3p+5}{4}.$$

Together with

$$0 \leqslant t \leqslant \frac{\sqrt{4p+1}-3}{4},$$

we get

$$\begin{aligned} 0 &< t^2 + t + \frac{3p+5}{4} \\ &\leqslant \left(\frac{\sqrt{4p+1}-3}{4}\right)^2 + \frac{\sqrt{4p+1}-3}{4} + \frac{3p+5}{4} \\ &= \frac{8p+9-\sqrt{4p+1}}{8} < p. \end{aligned}$$

It leads to a contradiction!

So the conclusion ③ holds for sufficiently large prime p, $p \equiv 1 \pmod 4$.

Finally, it remains to show that there are infinitely many of such $f(p)$. In fact,

$$p \mid f^2(p) + 1 \Rightarrow f(p) > \sqrt{p-1},$$

when $p \to \infty$, $f(p) \to \infty$. Therefore, $f(p)$ can take on infinitely many values.

As to the original problem, take $n = f(p)$, p is the corresponding prime divisor.

Second Day

0900 - 1330 July 17, 2008

4 Find all functions $f:(0, +\infty) \to (0, +\infty)$ (f is a function mapping positive real numbers to positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z satisfying $wx = yz$.

Solution Take

$$w = x = y = z = 1,$$

then we get $(f(1))^2 = f(1)$, so $f(1) = 1$.

For any real number $t > 0$, let $w = t$, $x = 1$, $y = z = \sqrt{t}$, we get

$$\frac{(f(t))^2 + 1}{2f(t)} = \frac{t^2 + 1}{2t},$$

which implies $(tf(t) - 1)(f(t) - t) = 0$.

So, for any $t > 0$,

$$f(t) = t \quad \text{or} \quad f(t) = \frac{1}{t}. \qquad ①$$

Suppose there exist b, $c \in (0, +\infty)$ such that $f(b) \neq b$, $f(c) \neq \frac{1}{c}$. By ①, we get b, c different from 1 and $f(b) = \frac{1}{b}$, $f(c) = c$.

Take $w = b$, $x = c$, $y = z = \sqrt{bc}$, then

$$\frac{\frac{1}{b^2} + c^2}{2f(bc)} = \frac{b^2 + c^2}{2bc},$$

i.e. $f(bc) = \dfrac{c + b^2c^3}{b(b^2 + c^2)}$.

By ①, $f(bc) = bc$ or $f(bc) = \frac{1}{bc}$. If $f(bc) = bc$, then

$$bc = \frac{c + b^2c^3}{b(b^2 + c^2)},$$

which yields $b^4c = c$, $b = 1$. Contradiction!

If $f(bc) = \frac{1}{bc}$, then

$$\frac{1}{bc} = \frac{c + b^2c^3}{b(b^2 + c^2)},$$

that yields $b^2c^4 = b^2$, $c = 1$. Contradiction!

Therefore, only two functions: $f(x) = x$, $x \in (0, +\infty)$ or $f(x) = \frac{1}{x}$, $x \in (0, +\infty)$. It is easy to verify that these two functions satisfy the given conditions.

5 Let n and k be positive integers with $k \geqslant n$ and $k - n$ an even number. Let $2n$ lamps labelled $1, 2, \cdots, 2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched from on to off or from off to on.

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

Determine the ratio $\frac{N}{M}$.

Solution The ratio is 2^{k-n}.

Lemma *For any positive integer t, call a t-element array $(a_1, a_2, \cdots, a_t)$ which consists of 0, 1 ($a_1, a_2, \cdots, a_t \in \{0, 1\}$) "good" if there are odd '0's in it. Prove that there are 2^{t-1} "good" arrays.*

Proof: In fact, for the same a_1, a_2, $\cdots$, a_t, when a_t is 0 or 1, the parity of 0s in these two arrays is different, so only one of the arrays is "good". There are 2^t array in all, we can match one array to the other, there are only a_t of them are different. Only one of these two arrays is "good". So of all the possible arrays, only half of them are "good". The lemma is proved.

Let A be the set of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

Let B the set of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

For any b in B, match all a in A to b if "a's elements are the same as b's (mod n)" (For example, take $n=2$, $k=4$, if $b=(2, 2, 2, 1)$, then it could correspond to $a=(4, 4, 2, 1)$, $a=(2, 2, 2, 1)$, $a=(2, 4, 4, 1)$etc). Since b in B, the number of 1, 2, $\cdots$, n must be odd; a in A, the number of 1, 2, $\cdots$, n of must be odd, and the number of $n+1$, $\cdots$, $2n$ must be even.

For any $i \in \{1, 2, \cdots, n\}$, if number of '$i$'s in b is b_i, then a only need to satisfy: for the positions taken by i in b the corresponding positions taken by i or $n+i$ in a, and the number of 'i' s is odd (thus the number of $n+i$ is even). By lemma and the product principle, there are $\prod_{i=1}^{n} 2^{b_i-1} = 2^{k-n}$ 'a' s corresponding to b, but only one of b (letting every position of a be the remainder when divided by n) in B corresponds to each a in A.

Therefore $|A| = 2^{k-n}|B|$, i.e. $N = 2^{k-n}M$.

Obviously $M \neq 0$ (because the sequence (1, 2, $\cdots$, n,

$n, \cdots, n) \in B$), so

$$\frac{N}{M} = 2^{k-n}.$$

6 Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA extended beyond A and to the ray BC extended beyond C, which is also tangent to the lines AD and CD.

Prove that the common external tangents to ω_1 and ω_2 intersects at a point on ω.

Proof Lemma 1 *Let ABCD be a convex quadrilateral, a circle ω tangent to ray BA extended beyond A and to the ray BC extended beyond C (as shown in Fig. 1), which is also tangent to the lines AD and CD. Then $AB + AD = CB + CD$.*

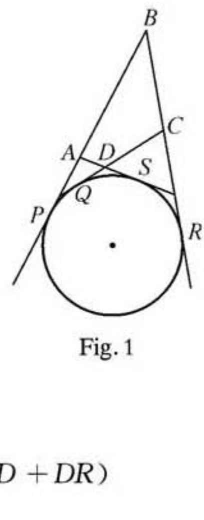


Fig. 1

Now we prove the Lemma 1.

Let ω meet AB, BC, CD, DA at P, Q, R, S respectively. As shown in Fig. 1,

$$\begin{aligned} & AB + AD = CB + CD \\ \Leftrightarrow & AB + (AD + DS) = CB + (CD + DR) \\ \Leftrightarrow & AB + AS = CB + CR \\ \Leftrightarrow & AB + AP = CB + CR \\ \Leftrightarrow & BP = BR. \end{aligned}$$

This ends the proof of Lemma 1.

Lemma 2 *If the radii of three circles $\odot O_1$, $\odot O_2$, $\odot O_3$ differ from each other, then their the homothetic centers are* *collinear*.

Now we prove Lemma 2.

Let X_3 be the homothetic center of $\odot O_1$ and $\odot O_2$, X_2 be the homothetic center of $\odot O_1$ and $\odot O_3$, and X_1 be the homothetic center of $\odot O_2$ and $\odot O_3$, r_i be the radius $\odot O_i$ $(i=1, 2, 3)$. By the property of homothetic

$$\frac{\overline{O_1X_3}}{\overline{X_3O_2}}=-\frac{r_1}{r_2}.$$

Here $\overline{O_1X_3}$ denotes the directed segment O_1X_3, as shown in Fig. 2. Similarly,

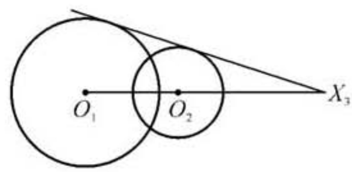


Fig. 2

$$\frac{\overline{O_2X_1}}{\overline{X_1O_3}}=-\frac{r_2}{r_3},\ \frac{\overline{O_3X_2}}{\overline{X_2O_1}}=-\frac{r_3}{r_1}.$$

So

$$\frac{\overline{O_1X_3}}{\overline{X_3O_2}}\cdot\frac{\overline{O_2X_1}}{\overline{X_1O_3}}\cdot\frac{\overline{O_3X_2}}{\overline{X_2O_1}}=\left(-\frac{r_1}{r_2}\right)\left(-\frac{r_2}{r_3}\right)\left(-\frac{r_3}{r_1}\right)=-1.$$

By *Menelaus* Theorem, X_1, X_2, X_3 are collinear.

Let ω_1, ω_2 meet AC at U, V respectively. As shown in Fig. 3,

$$\begin{aligned}AV&=\frac{AD+AC-CD}{2}=\frac{AC}{2}+\frac{AD-CD}{2}\\&=\frac{AC}{2}+\frac{CB-AB}{2}=\frac{AC+CB-AB}{2}\\&=CU.\end{aligned}$$

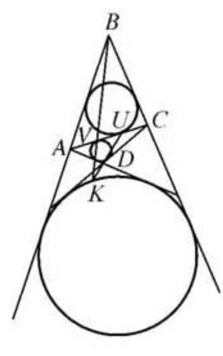


Fig. 3

Hence the excircle ω_3 of $\triangle ABC$ on the side AC meet AC at V. Therefore ω_2, ω_3 meet at point V, i. e. V is the homothetic center of ω_2, ω_3. Denote the homothetic centres of ω_1, ω_2 by K (i. e. K is the intersection of two external common tangents to ω_2, ω_3), by Lemma 2, K, V, B are collinear.

Similarly, K, D, U are collinear.

Since $BA \neq BC$, then $U \neq V$ (Otherwise, by $AV = CU$, we know $U = V$ is the midpoint of side AC. It is contradictory to $BA \neq BC$). So BV does not coincide with DU, i. e. $K = BV \cap DU$.

We now prove that K is on the circle ω.

Construct a tangent l of ω which is parallel to AC. Let l meet ω at T. We will show that B, V, T are collinear.

As shown in Fig. 4, l intersects BA, BC at A_1, C_1 respectively, then ω is the excircle of $\triangle BA_1C_1$ on the side A_1C_1, meet A_1C_1 at T, meanwhile ω_3 is the excircle of $\triangle ABC$ on the side AC, meet AC at V. Since $AC \parallel A_1C_1$, B is the homothetic center of $\triangle BAC$ and $\triangle BA_1C_1$, V and T are corresponding points, so B, V, T are collinear.

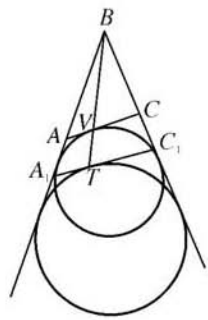


Fig. 4

Similarly, D, V, T are collinear. This means $K = T$.

This ends the proof.